Logic

Logics on words

- Regular expressions give operational descriptions of regular languages.
- Often the natural description of a language is declarative:
  - **even number of a's and even number of b's** vs.

$$\left(aa + bb + (ab + ba)(aa + bb)^*(ba + ab)\right)^*$$

  - **words not containing ‘hello’**
- Goal: find a declarative language able to express all the regular languages, and only the regular languages.

Logics on words

- Idea: use a logic that has an interpretation on words
- A formula expresses a property that each word may satisfy or not, like
  - **the word contains only a's**
  - **the word has even length**
  - **between every occurrence of an a and a b there is an occurrence of a c**
- Every formula (indirectly) defines a language: the language of all the words over the given fixed alphabet that satisfy it.

First-order logic on words

- Atomic formulas: for each letter a we introduce the formula $Q_a(x)$, with intuitive meaning: **the letter at position x is an a**.

First-order logic on words: Syntax

- Formulas constructed out of atomic formulas by means of standard "logic machinery":
  - Alphabet $\Sigma = \{a, b, \dots\}$ and position variables $V = \{x, y, \dots\}$
  - $Q_a(x)$ is a formula for every $a \in \Sigma$ and $x \in V$.
  - $x < y$ is a formula for every $x, y \in V$
  - If $\varphi, \varphi_1, \varphi_2$ are formulas then so are $\neg\varphi$ and $\varphi_1 \vee \varphi_2$
  - If φ is a formula then so is $\exists x\ \varphi$ for every $x \in V$

Abbreviations

- $\varphi_1 \wedge \varphi_2 \coloneqq \neg(\neg \varphi_1 \vee \neg\varphi_2)$
- $\varphi_1 \rightarrow \varphi_2 \coloneqq \neg\varphi_1 \vee \varphi_2$
- $\varphi_1 \leftrightarrow \varphi_2 \coloneqq (\varphi_1 \wedge \varphi_2) \vee (\neg\varphi_1 \wedge \neg\varphi_2)$
- $\forall x\, \varphi \coloneqq \neg\, \exists x\, \neg\varphi$

Abbreviations

- $\mathrm{first}(x) \coloneqq \neg\exists y\ \ y < x$ $\qquad$ $\mathrm{last}(x) \coloneqq \neg\exists y\ \ x < y$
- $y = x + 1 \ \coloneqq x < y \wedge \neg\exists z\,(x < z \wedge z < y)$
- $y = x + 2 \ \coloneqq \exists z\,(z = x + 1 \wedge y = z + 1)$

  …
- $y = x + k \ \coloneqq \exists z\,(z = x + 1 \wedge y = z + (k - 1))$

- $x < k \ \coloneqq \forall y \forall z\,(\mathrm{first}(y) \wedge z = y + k) \to x < z)$
- $\mathrm{last} < k \coloneqq \forall x\,(\mathrm{last}(x) \to x < k)$

Examples (without semantics yet)

- “The last letter is a b and before it there are only a’s.”
- “Every a is immediately followed by a b.”
- “Every a is immediately followed by a b, unless it is the last letter.”
- “Between every a and every later b there is a c.”

Examples (without semantics yet)

- “The last letter is a b and before it there are only a’s.”

$$\exists x\; Q_b(x) \wedge \forall x\, (\mathrm{last}(x) \rightarrow Q_b(x) \;\wedge\; \neg\mathrm{last}(x) \rightarrow Q_a(x))$$

- “Every a is immediately followed by a b.”

- “Every a is immediately followed by a b, unless it is the last letter.”

- “Between every a and every later b there is a c.”

Examples (without semantics yet)

- “The last letter is a *b* and before it there are only *a*’s.”

$$\exists x\, Q_b(x) \wedge \forall x\,(\mathrm{last}(x) \rightarrow Q_b(x) \;\wedge\; \neg\mathrm{last}(x) \rightarrow Q_a(x))$$

- “Every *a* is immediately followed by a *b*.”

$$\forall x\,(Q_a(x) \rightarrow \exists y\,(y = x + 1 \wedge Q_b(y)))$$

- “Every *a* is immediately followed by a *b*, unless it is the last letter.”

- “Between every *a* and every later *b* there is a *c*.”

Examples (without semantics yet)

- "The last letter is a b and before it there are only a's."

$$\exists x\, Q_b(x) \land \forall x\, (\mathrm{last}(x) \to Q_b(x) \;\land\; \neg\mathrm{last}(x) \to Q_a(x))$$

- "Every a is immediately followed by a b."

$$\forall x\, (Q_a(x) \to \exists y\, (y = x + 1 \land Q_b(y)))$$

- "Every a is immediately followed by a b, unless it is the last letter."

$$\forall x\, (Q_a(x) \to \forall y\, (y = x + 1 \to Q_b(y)))$$

- "Between every a and every later b there is a c."

Examples (without semantics yet)

- “The last letter is a b and before it there are only a’s.”

$$\exists x\, Q_b(x) \wedge \forall x\, (\mathrm{last}(x) \rightarrow Q_b(x) \;\wedge\; \neg\mathrm{last}(x) \rightarrow Q_a(x))$$

- “Every a is immediately followed by a b.”

$$\forall x\, (Q_a(x) \rightarrow \exists y\, (y = x + 1 \wedge Q_b(y)))$$

- “Every a is immediately followed by a b, unless it is the last letter.”

$$\forall x\, (Q_a(x) \rightarrow \forall y\, (y = x + 1 \rightarrow Q_b(y)))$$

- “Between every a and every later b there is a c.”

$$\forall x \forall y\, (Q_a(x) \wedge Q_b(y) \wedge x < y \rightarrow \exists z\, (x < z \wedge z < y \wedge Q_c(z)))$$

First-order logic on words: Semantics

- Formulas are interpreted on pairs $(w, \mathcal{J})$ called interpretations, where
  - w is a word, and
  - $\mathcal{J}$ assigns positions to the free variables of the formula (and maybe to others too—who cares)
- It does not make sense to say a formula is true or false: it can only be true or false for a given interpretation.
- If the formula has no free variables (if it is a sentence), then for each word it is either true or false.

- Satisfaction relation:

$$\begin{array}{lllll}
(w, \mathfrak{I}) & \models & Q_a(x) & \mathit{iff} & w[\mathfrak{I}(x)] = a \\
(w, \mathfrak{I}) & \models & x < y & \mathit{iff} & \mathfrak{I}(x) < \mathfrak{I}(y) \\
(w, \mathfrak{I}) & \models & \neg\varphi & \mathit{iff} & (w, \mathfrak{I}) \not\models \varphi \\
(w, \mathfrak{I}) & \models & \varphi 1 \vee \varphi_2 & \mathit{iff} & (w, \mathfrak{I}) \models \varphi_1 \ \mathit{or} \ (w, \mathfrak{I}) \models \varphi_2 \\
(w, \mathfrak{I}) & \models & \exists x\, \varphi & \mathit{iff} & |w| \geq 1 \ \mathit{and\ some} \ i \in \{1, \ldots, |w|\} \\
 & & & & \mathit{satisfies} \ (w, \mathfrak{I}[i/x]) \models \varphi
\end{array}$$

- More logic jargon:
  - A formula is valid if it is true for all its interpretations
  - A formula is satisfiable if is is true for at least one of its interpretations

The empty word ...

- ... satisfies all universally quantified formulas, and no existentially quantified formula.

Can FOL express non-regular languages? Can FOL express all regular languages?

- The language $L(\varphi)$ of a sentence φ is the set of words that satisfy φ.
- A language L is expressible in first-order logic or FO-definable if some sentence φ satisfies $L(\varphi) = L$.
- Proposition: a language over a one-letter alphabet is expressible in first-order logic iff it is finite or co-finite (its complement is finite).
- Consequence: we can only express regular languages, but not all, not even the language of words of even length.

Proof sketch

1. If L is finite, then it is FO-definable

2. If L is co-finite, then it is FO-definable.

Proof sketch

3. If L is FO-definable (over a one-letter alphabet), then it is finite or co-finite.
    1) We define a new logic QF (quantifier-free fragment)
    2) We show that a language is QF-definable iff it is finite or co-finite
    3) We show that a language is QF-definable iff it is FO-definable.

1) The logic QF

- $x < k$ $\quad x > k$
  $x < y + k$ $\quad x > y + k$
  $k <$ last $\quad k >$ last
  are formulas for every variable x, y and every $k \geq 0$.
- If f_1, f_2 are formulas, then so are $f_1 \vee f_2$ and $f_1 \wedge f_2$

2) L is QF-definable iff it is finite or co-finite

(→) Let f be a sentence of QF.

Then f is a positive boolean combination of formulas $k < \text{last}$ and $k > \text{last}$.

$L(k < \text{last}) = \{k+1, k+2, \dots\}$ is co-finite (we identify words and numbers)

$L(k > \text{last}) = \{0, 1, \dots, k\}$ is finite

$L(f_1 \vee f_2) = L(f_1) \cup L(f_2)$ and so if $L(f_1)$ and $L(f_2)$ finite or co-finite then L is finite or co-finite.

$L(f_1 \wedge f_2) = L(f_1) \cap L(f_2)$ and so if $L(f_1)$ and $L(f_2)$ finite or co-finite then L is finite or co-finite.

2) L is QF-definable iff it is finite or co-finite

(←) If $L = \{k_1, \ldots, k_n\}$ is finite, then

$$(k_1 - 1 < \text{last} \wedge \text{last} < k_1 + 1) \vee \cdots \vee$$
$$(k_n - 1 < \text{last} \wedge \text{last} < k_n + 1)$$

expresses L.

If L is co-finite, then its complement is finite, and so expressed by some formula. We show that for every f some formula $\text{neg}(f)$ expresses $\overline{L(f)}$

- $\text{neg}(k < \text{last}) = (k - 1 < \text{last} \wedge \text{last} < k + 1) \vee \text{last} < k$
- $\text{neg}(f_1 \vee f_2) = \text{neg}(f_1) \wedge \text{neg}(f_2)$
- $\text{neg}(f_1 \wedge f_2) = \text{neg}(f_1) \vee \text{neg}(f_2)$

3) Every first-order formula φ has an equivalent QF-formula $QF(\varphi)$

- $QF(x < y) = x < y + 0$
- $QF(\neg\varphi) = \text{neg}\big(QF(\varphi)\big)$
- $QF(\varphi_1 \vee \varphi_2) = QF(\varphi_1) \vee QF(\varphi_2)$
- $QF(\varphi_1 \wedge \varphi_2) = QF(\varphi_1) \wedge QF(\varphi_2)$
- $QF(\exists x\, \varphi) =$
  - Put $QF(\varphi)$ in disjunctive normal form. Assume $QF(\varphi)$= $(\varphi_1 \vee \ldots \vee \varphi_n)$, where each φ_i is a conjunction of atomic formulas.
  - Since ∃x $(\varphi_1 \vee \ldots \vee \varphi_n) \equiv$ ∃x $\varphi_1 \vee \ \ldots \ \vee$ ∃x φ_n, it suffices to define $QF(\exists x\, \varphi)$ for the case in which φ is a conjunction of atomic formulas of QF
  - For this case, see example in the next slide.

- Consider the formula

$$\begin{aligned} \exists x \quad & x < y + 3 \quad \wedge \\ & z < x + 4 \quad \wedge \\ & z < y + 2 \quad \wedge \\ & y < x + 1 \end{aligned}$$

- The equivalent QF-formula is

$$z < y + 8 \ \wedge \ y < y + 5 \ \wedge \ z < y + 2$$

Monadic second-order logic

- First-order variables: interpreted on positions
- Monadic second-order variables: interpreted on sets of positions.
  - Diadic second-order variables: interpreted on relations over positions
  - Monadic third-order variables: interpreted on sets of sets of positions
  - New atomic formulas: $x \in X$

Expressing „even length“

- Express

  There is a set X of positions such that

  – X contains exactly the even positions, and

  – the last position belongs to X.

- Express

  X contains exactly the even positions

  as

A position is in X iff it is the second position or the second successor of another position of X

Syntax and semantics of MSO

- New set $\{X, Y, Z, \ldots\}$ of second-order variables
- New syntax: $x \in X$ and $\exists X\ \varphi$
- New semantics:
  - Interpretations now also assign sets of positions to the free second-order variables.
  - Satisfaction defined as expected.

Expressing „even length“

- $\mathrm{second}(x) = \exists y\ (\mathrm{first}(y) \wedge x = y + 1)$

- $\mathrm{Even}(X) = \forall y \left(x \in X \leftrightarrow \left(\begin{matrix} \mathrm{second}(x) \\ \vee\ \exists y\ (x = y + 2 \wedge y \in X) \end{matrix} \right) \right)$

- $\mathrm{Evenlength} = \exists X \left(\begin{matrix} \mathrm{Even}(X) \wedge \\ \forall x\ (\mathrm{last}(x) \to x \in X) \end{matrix} \right)$

Expressing $c^*(ab)^*d^*$

- Express:

  There is a block X of consecutive positions such that

  - **before X there are only c‘s;**
  - **after X there are only d‘s;**
  - **a‘s and b‘s alternate in X;**
  - **the first letter in X is an a, and the last is a b.**

- Then we can take the formula

  $$\exists X\ (Cons(X)\ \wedge Boc(X) \wedge Aod(X) \wedge Alt(X) \wedge Fa(X) \wedge Lb(X)\ )$$

- **X is a block of consecutive positions**

- **Before X there are only c‘s**

- **In X a‘s and b‘s alternate**

- X is a block of consecutive positions

$$\mathrm{Cons}(X) := \forall x \in X\ \forall y \in X\ \ (x < y \rightarrow (\forall z\, (x < z \wedge z < y) \rightarrow z \in X))$$

- Before X there are only c‘s

- In X a‘s and b‘s alternate

- X is a block of consecutive positions

$$\mathrm{Cons}(X) := \forall x \in X\ \forall y \in X\ \ (x < y \to (\forall z\ (x < z \land z < y) \to z \in X))$$

- Before X there are only c‘s

$$\mathrm{Before}(x, X) := \forall y \in X\ x < y$$

$$\mathrm{Before\_only\_c}(X) := \forall x\ \mathrm{Before}(x, X) \to Q_c(x)$$

- In X a‘s and b‘s alternate

- X is a block of consecutive positions

$$\mathrm{Cons}(X) := \forall x \in X\ \forall y \in X\ \ (x < y \to (\forall z\ (x < z \land z < y) \to z \in X))$$

- Before X there are only c‘s

$$\mathrm{Before}(x, X) := \forall y \in X\ x < y$$

$$\mathrm{Before\_only\_c}(X) := \forall x\ \mathrm{Before}(x, X) \to Q_c(x)$$

- In X a‘s and b‘s alternate

$$\begin{aligned}\mathrm{Alternate}(X) := \ \forall x \in X\ \ (\ & Q_a(x) \to \forall y \in X\ (y = x + 1 \to Q_b(y)\ ) \\ & \land \\ & Q_b(x) \to \forall y \in X\ (y = x + 1 \to Q_a(y)\ )\ )\end{aligned}$$

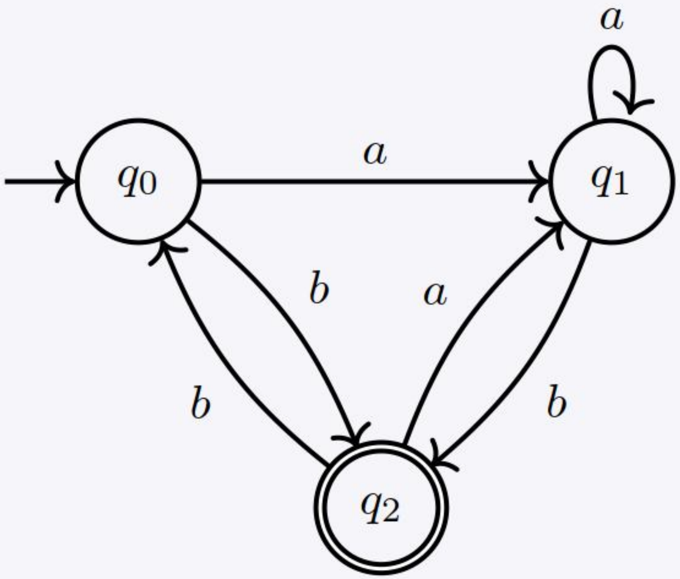
Every regular language is expressible in MSO logic

- Goal: given an arbitrary regular language L, construct an MSO sentence φ s.t. $L = L(\varphi)$.
- It suffices to construct φ s.t. $w \in L$ iff $w \in L(\varphi)$ for every nonempty word w. (Avoid the corner-case of the empty word.)
- We use: if L is regular, then there is a DFA A recognizing L.
- Idea: construct a formula expressing

the run of A on this word is accepting

- Fix a regular language L.
- Fix a DFA A with states $q_0, \dots, q_n$ recognizing L.
- Fix a nonempty word $w = a_1 a_2 \dots a_m$.
- Let $R(q)$ be the set of positions i such that after reading $a_1 a_2 \dots a_i$ the automaton A is in state q.
- We have:

A accepts w iff $m \in P_q$ for some final state q.



Run: $q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \xrightarrow{b} q_2 \xrightarrow{b} q_0 \xrightarrow{b} q_2$

Position: 1 2 3 4 5

$$R_w(q_0) = \{4\}$$
$$R_w(q_1) = \{1, 2\}$$
$$R_w(q_2) = \{3, 5\}$$

- Assume we can construct a formula

$$\text{Visits}(X_0, \dots, X_n)$$

which is true for $(w, \mathcal{I})$ iff

$$\mathcal{I}(X_0) = R(q_0), \dots, \mathcal{I}(X_n) = R(q_n)$$

- Then $(w, \mathcal{I})$ satisfies the formula

$$\forall X_0 \cdots \forall X_n \, \forall x \left((\text{Visits}(X_0, \dots, X_n) \land \text{last}(x)) \to \bigvee_{q_i \in F} x \in X_i \right)$$

iff the state after the last position is accepting, and we easily get a formula expressing L .

- To construct Visits$(X_0, \ldots, X_n)$ we observe that the sets $R(q)$ are the unique sets satisfying
  a) $1 \in R(\delta(q_0, a_1))$ i.e., after reading the first letter the DFA is in state $\delta(q_0, a_1)$.
  b) The sets $R(q)$ build a partition of the set of positions, i.e., the DFA is always in exactly one state.
  c) If $i \in R(q)$ and $\delta(q, a_{i+1}) = q'$ then $i + 1 \in R(q')$, i.e., the sets „match“ δ.
- We give formulas for a) , b), and c)

$$\mathrm{Init}(X_0, \ldots, X_n) = \exists x \left(\mathrm{first}(x) \land \left(\bigvee_{a \in \Sigma} (Q_a(x) \land x \in X_{i_a}) \right) \right)$$

$$\mathrm{Partition}(X_0, \ldots, X_n) = \forall x \left(\bigvee_{i=0}^{n} x \in X_i \;\land\; \bigwedge_{\substack{i,\, j = 0 \\ i \neq j}}^{n} (x \in X_i \rightarrow x \notin X_j) \right)$$

- Formula for c)

$$\mathrm{Respect}(X_0, \ldots, X_n) =$$

$$\forall x \forall y \left(y = x + 1 \rightarrow \bigvee_{\substack{a \in \Sigma \\ i, j \in \{0, \ldots, n\} \\ \delta(q_i, a) = q_j}} (x \in X_i \wedge Q_a(x) \wedge y \in X_j) \right)$$

- Together:

$$\begin{aligned} \mathrm{Visits}(X_0, \ldots X_n) := {} & \mathrm{Init}(X_0, \ldots, X_n) \wedge \\ & \mathrm{Partition}(X_0, \ldots, X_n) \wedge \\ & \mathrm{Respect}(X_0, \ldots, X_n) \end{aligned}$$

Every language expressible in MSO logic is regular

- Recall: an interpretation of a formula is a pair $(w, \mathcal{I})$ consisting of a word w and assignments $\mathcal{I}$ to the free first and second order variables (and perhaps to others).

$$\left(aab\ ,\ \begin{array}{l} x \mapsto 1 \\ y \mapsto 3 \\ X \mapsto \{2,3\} \\ Y \mapsto \{1,2\} \end{array} \right) \qquad \left(ba\ ,\ \begin{array}{l} x \mapsto 2 \\ y \mapsto 1 \\ X \mapsto \emptyset \\ Y \mapsto \{1\} \end{array} \right)$$

- We encode interpretations as words.

$$\left(aab\ ,\ \begin{array}{l} x \mapsto 1 \\ y \mapsto 3 \\ X \mapsto \{2,3\} \\ Y \mapsto \{1,2\} \end{array} \right) \qquad \left(ba\ ,\ \begin{array}{l} x \mapsto 2 \\ y \mapsto 1 \\ X \mapsto \emptyset \\ Y \mapsto \{1\} \end{array} \right)$$

$$\begin{array}{cccc} & a & a & b \\ x & 1 & 0 & 0 \\ y & 0 & 0 & 1 \\ X & 0 & 1 & 1 \\ Y & 1 & 1 & 0 \end{array} \qquad \begin{array}{ccc} & b & a \\ x & 0 & 1 \\ y & 1 & 0 \\ X & 0 & 0 \\ Y & 1 & 0 \end{array}$$

- Given a formula with n free variables, we encode an interpretation $(w, \mathcal{I})$ as a word $enc(w, \mathcal{I})$ over the alphabet $\Sigma \times \{0,1\}^n$.

- The language of the formula φ , denoted by $L(\varphi)$, is given by

$$L(\varphi) := \{enc(w, \mathcal{I}) \mid (w, \mathcal{I}) \models \varphi\}$$

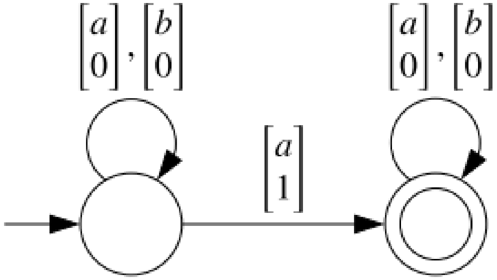
- We prove by induction on the structure of φ that $L(\varphi)$ is regular (and explicitely construct an automaton for it).

Case $\varphi = Q_a(x)$

- $\varphi = Q_a(x)$. Then $free(\varphi) = x$, and the interpretations of φ are encoded as words over $\Sigma \times \{0, 1\}$. The language $L(\varphi)$ is given by

$$L(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \begin{matrix} \dots \\ \dots \end{matrix} \begin{bmatrix} a_k \\ b_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i \in \{0, 1\} \text{ for every } i \in \{1, \dots, k\}, \text{ and} \\ b_i = 1 \text{ for exactly one index } i \in \{1, \dots, k\} \text{ such that } a_i = a \end{array} \right\}$$

and is recognized by

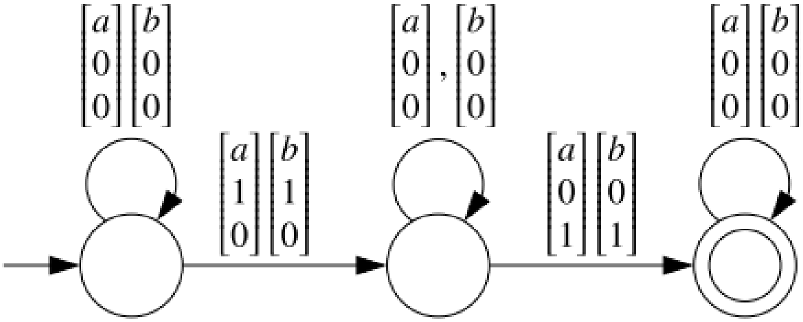


Case $\varphi = x < y$

- $\varphi = x < y$. Then $free(\varphi) = \{x, y\}$, and the interpretations of ϕ are encoded as words over $\Sigma \times \{0, 1\}^2$. The language $L(\varphi)$ is given by

$$L(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \begin{matrix} \\ \cdots \\ \cdots \end{matrix} \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i, c_i \in \{0, 1\} \text{ for every } i \in \{1, \ldots, k\}, \\ b_i = 1 \text{ for exactly one index } i \in \{1, \ldots, k\}, \\ c_j = 1 \text{ for exactly one index } j \in \{1, \ldots, k\}, \text{ and} \\ i < j \end{array} \right\}$$

and is recognized by

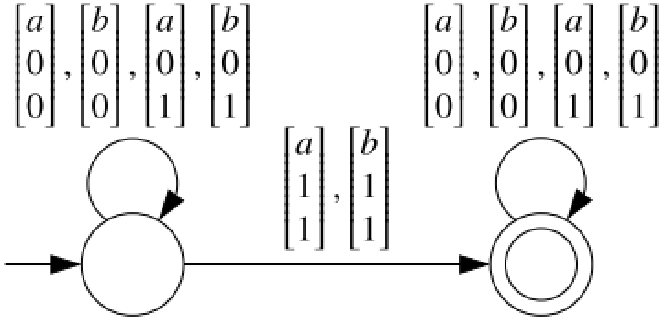


Case $\varphi = x \in X$

- $\varphi = x \in X$. Then $\mathit{free}(\varphi) = \{x, X\}$, and interpretations are encoded as words over $\Sigma \times \{0,1\}^2$. The language $L(\varphi)$ is given by

$$L(\varphi) = \left\{ \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \begin{matrix} \ldots \\ \ldots \end{matrix} \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} \;\middle|\; \begin{array}{l} k \geq 0, \\ a_i \in \Sigma \text{ and } b_i, c_i \in \{0,1\} \text{ for every } i \in \{1, \ldots, k\}, \\ b_i = 1 \text{ for exactly one index } i \in \{1, \ldots, k\}, \text{ and} \\ \text{for every } i \in \{1, \ldots, k\}, \text{ if } b_i = 1 \text{ then } c_i = 1 \end{array} \right\}$$

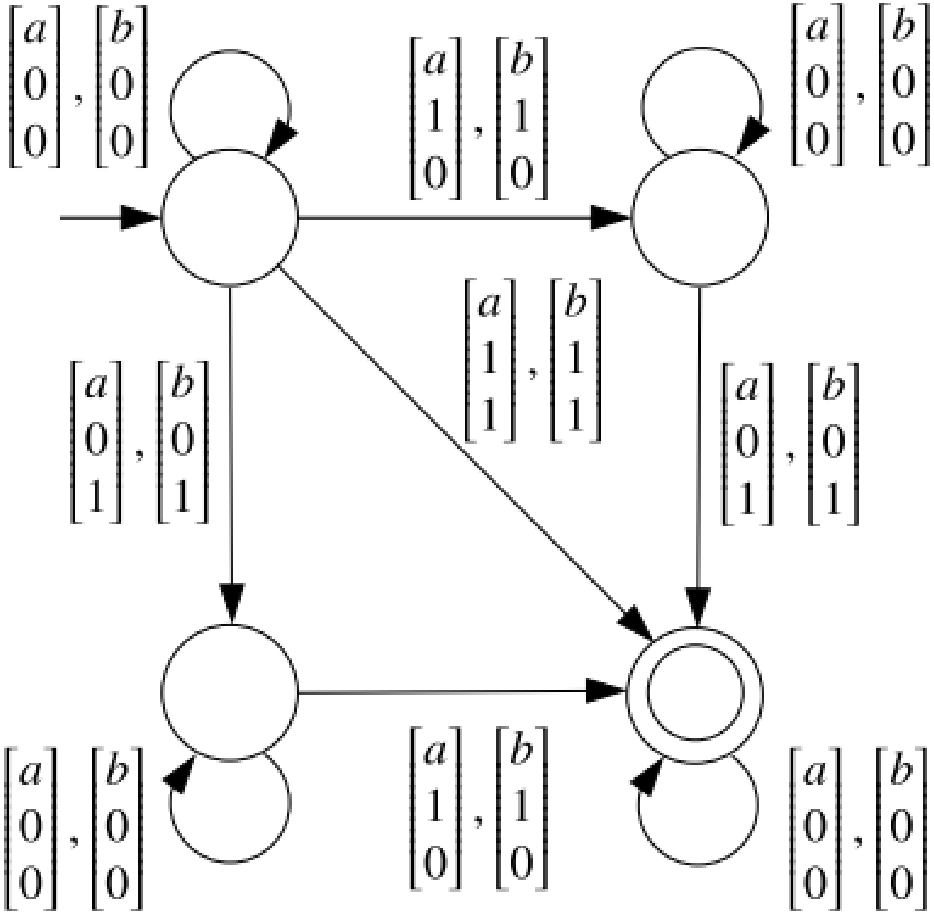
and is recognized by



Case $\varphi = \neg\psi$

- Then free(φ) = free(ψ) . By i.h. $L(\psi)$ is regular.
- $L(\varphi)$ is equal to $\overline{L(\psi)}$ minus the words that do not encode any implementation („the garbage“).
- Equivalently, $L(\varphi)$ is equal to the intersection of $\overline{L(\psi)}$ and the encodings of all interpretations of ψ.
- We show that the set of these encodings is regular.
  - Condition for encoding: Let x be a free first-oder variable of ψ . The projection of an encoding onto x must belong to 0^*10^* (because it represents one position).
  - So we just need an automaton for the words satisfying this condition for every free first-order variable.

Example: free(φ) = $\{x, y\}$

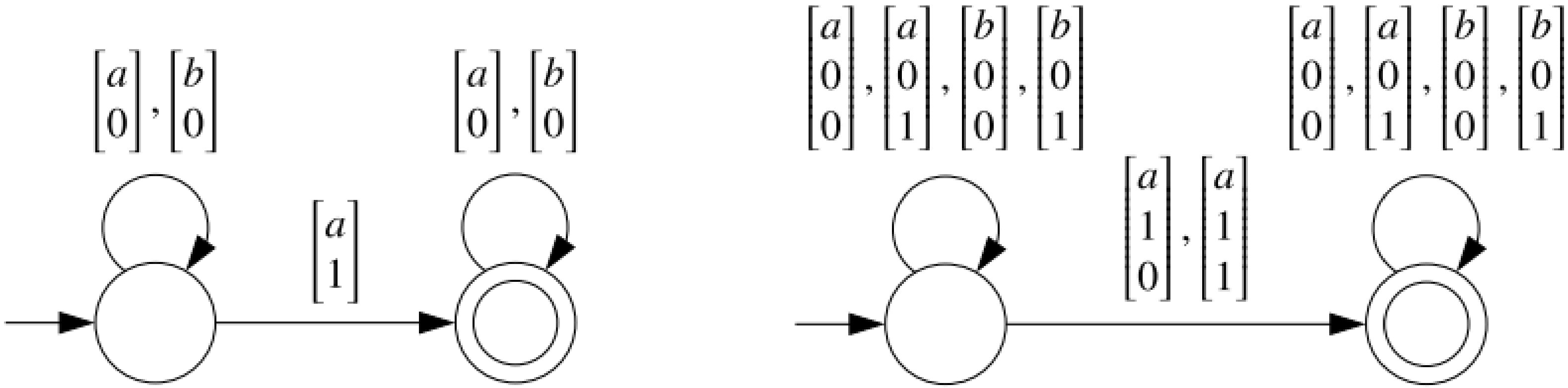


Case $\varphi = \varphi_1 \lor \varphi_2$

- Then $\text{free}(\varphi) = \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$. By i.h. $L(\varphi_1)$ and $L(\varphi_2)$ are regular.
- If $\text{free}(\varphi_1) = \text{free}(\varphi_2)$ then $L(\varphi) = L(\varphi_1) \cup L(\varphi_2)$ and so $L(\varphi)$ is regular.
- If $\text{free}(\varphi_1) \neq \text{free}(\varphi_2)$ then we extend $L(\varphi_1)$ to L_1 encoding all interpretations of $\text{free}(\varphi_1) \cup \text{free}(\varphi_2)$ whose projection onto $\text{free}(\varphi_1)$ belongs to $L(\varphi_1)$. Similarly we extend $L(\varphi_2)$ to L_2. We have
  - L_1 and L_2 are regular.
  - $L(\varphi) = L_1 \cup L_2$.

Example: $\varphi = Q_a(x) \vee Q_b(y)$

- L_1 contains the encodings of all interpretations $(w, \{x \mapsto n_1, y \mapsto n_2\})$ such that the encoding of $(w, \{x \mapsto n_1\})$ belongs to $L(Q_a(x))$.
- Automata for $L(Q_a(x))$ and L_1:



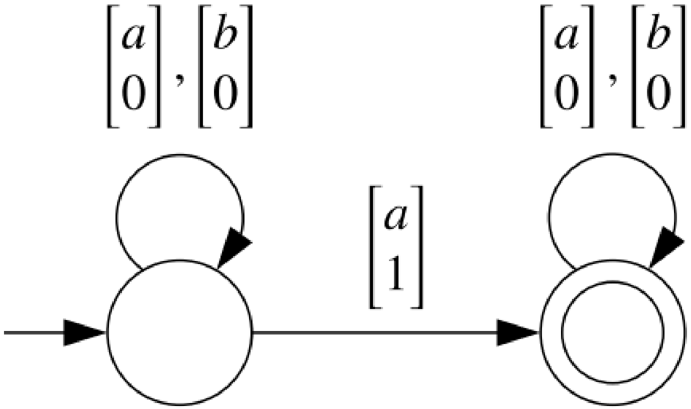
Cases $\varphi = \exists x\ \psi$ and $\varphi = \exists X\ \psi$

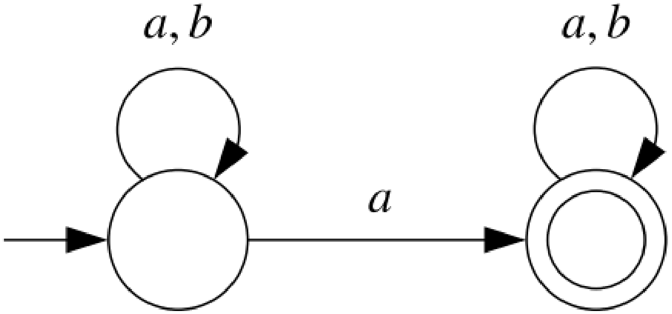
- Then free(φ)= free(ψ)$\setminus \{x\}$ or

  free(φ)= free(ψ)$\setminus \{X\}$
- By i.h. $L(\psi)$ is regular.
- $L(\varphi)$ is the result of projecting $L(\psi)$ onto the components for free(ψ)$\setminus \{x\}$ or for free(ψ)$\setminus \{X\}$.

Example: $\varphi = Q_a(x)$

- Automata for $Q_a(x)$ and $\exists x\, Q_a(x)$



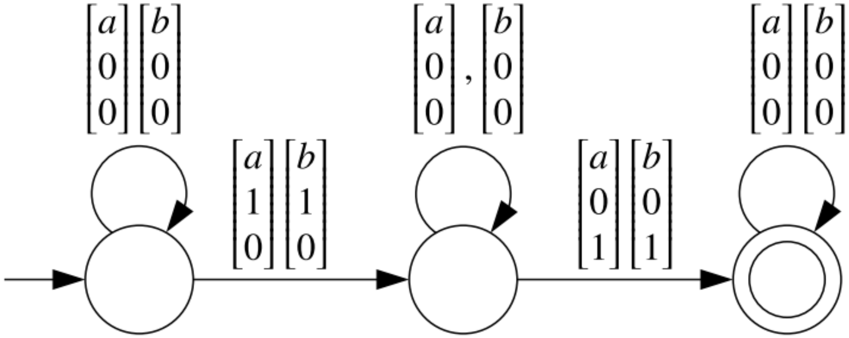


The mega-example

- We compute an automaton for
  $\exists x \, (\text{last}(x) \land Q_b(x)) \land \forall x \, (\neg \text{last}(x) \to Q_a(x))$
- First we rewrite it into
  $\exists x \, (\text{last}(x) \land Q_b(x)) \land \neg \exists x \, (\neg \text{last}(x) \land \neg Q_a(x))$
- In the next slides we
  1. compute a DFA for last(x)
  2. compute DFAs for $\exists x \, (\text{last}(x) \land Q_b(x))$ and $\neg \exists x \, (\neg \text{last}(x) \land \neg Q_a(x))$
  3. compute a DFA for the complete formula.
- We denote the DFA for a formula ψ by $[\psi]$.

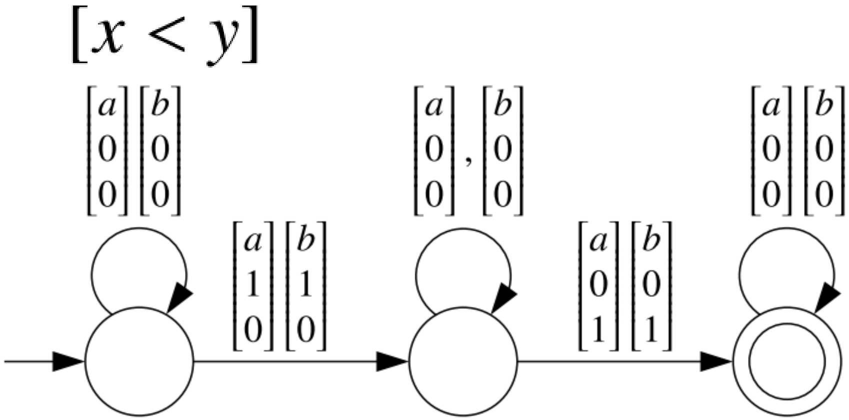
[last(x)]

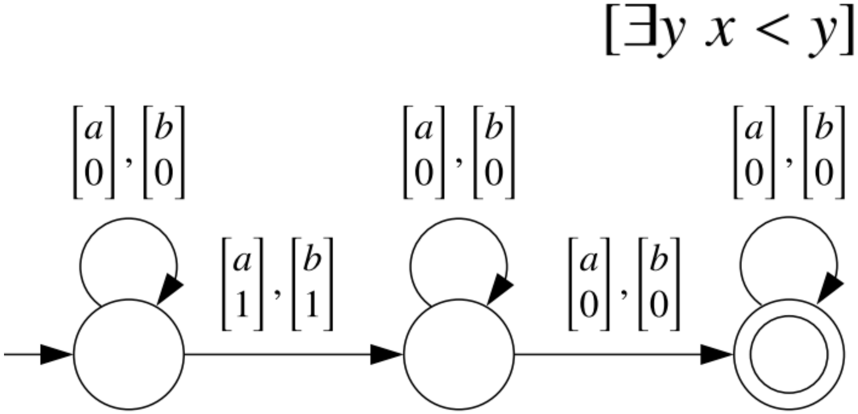
$[x < y]$



$\begin{bmatrix}a\\0\\0\end{bmatrix}\begin{bmatrix}b\\0\\0\end{bmatrix}$
$\begin{bmatrix}a\\1\\0\end{bmatrix}\begin{bmatrix}b\\1\\0\end{bmatrix}$
$\begin{bmatrix}a\\0\\0\end{bmatrix},\begin{bmatrix}b\\0\\0\end{bmatrix}$
$\begin{bmatrix}a\\0\\1\end{bmatrix}\begin{bmatrix}b\\0\\1\end{bmatrix}$
$\begin{bmatrix}a\\0\\0\end{bmatrix}\begin{bmatrix}b\\0\\0\end{bmatrix}$

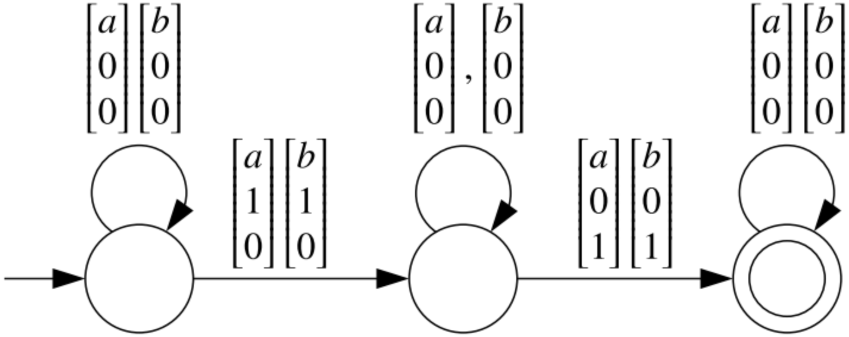
$[\text{last}(x)]$


$[x < y]$

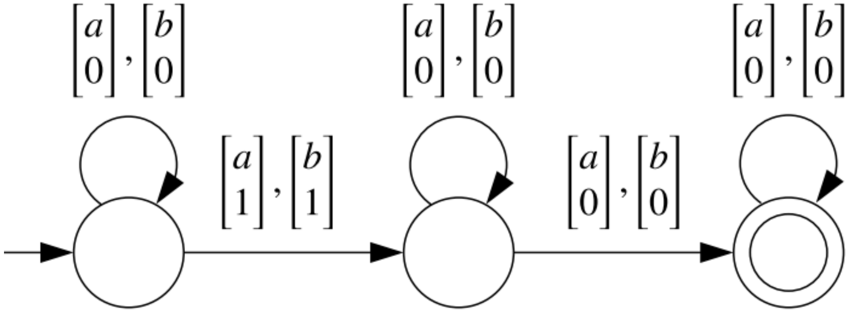

$[\exists y\ x < y]$

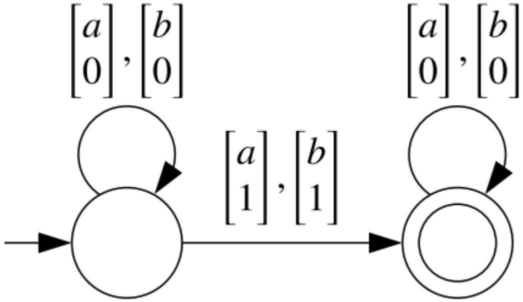
$[\mathrm{last}(x)]$

$[x < y]$

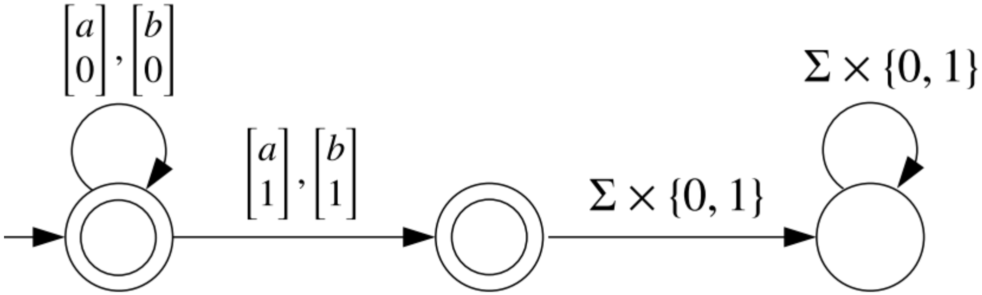


$[\exists y\ x < y]$



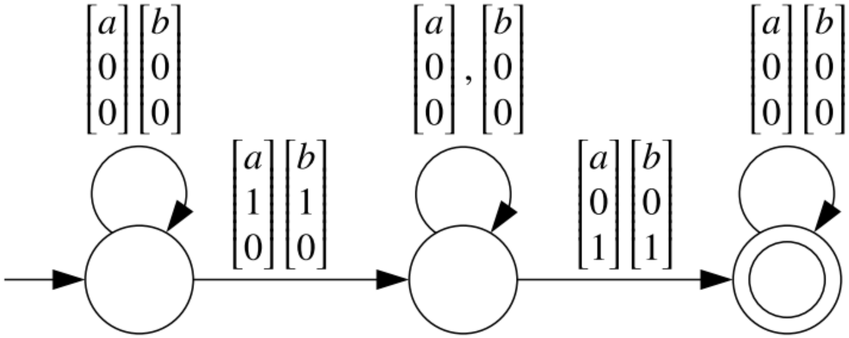


$Enc(\exists y\ x < y)$

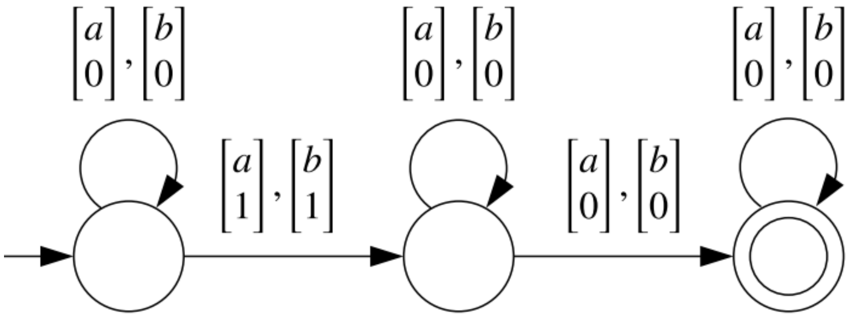


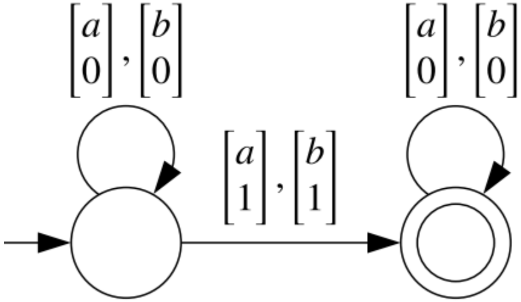
[last(x)]

[$x < y$]

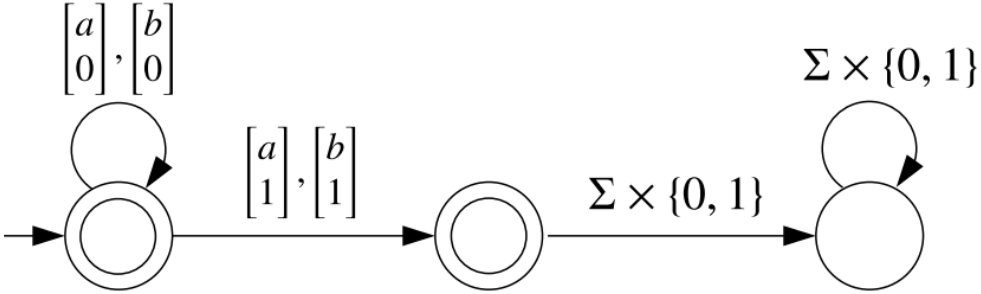


[$\exists y\ x < y$]

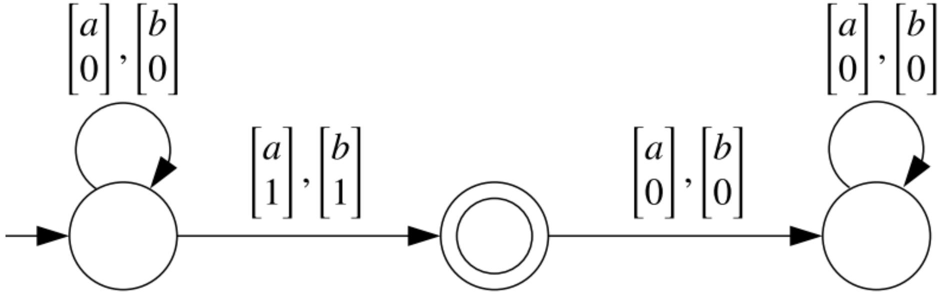


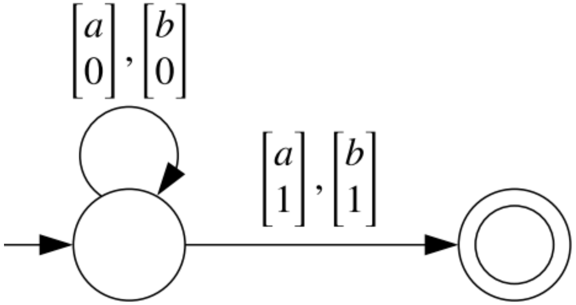


$Enc(\exists y\ x < y)$



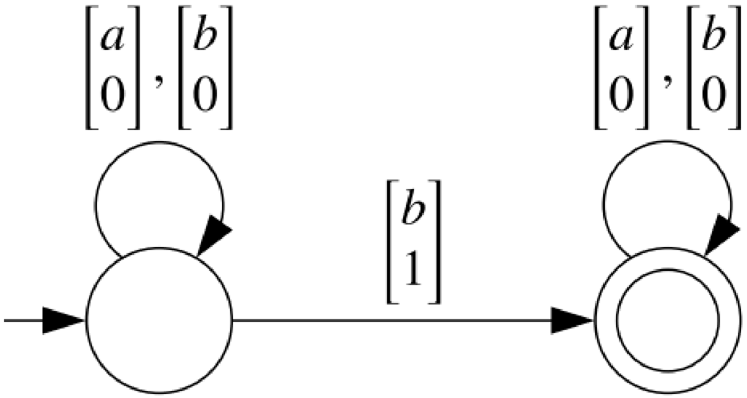
[$\overline{\exists y\ x < y}$]



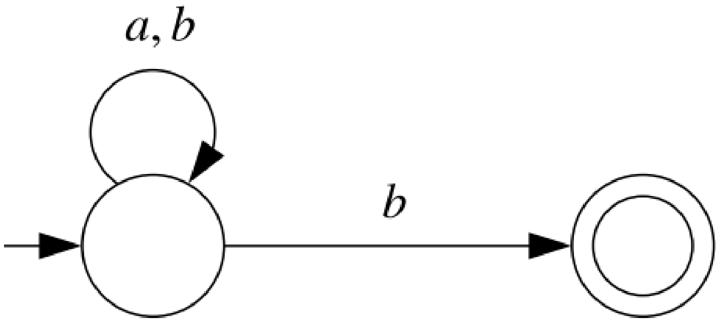


[last(x)]

$[\exists x\,(\text{last}(x) \wedge Q_b(x))]$

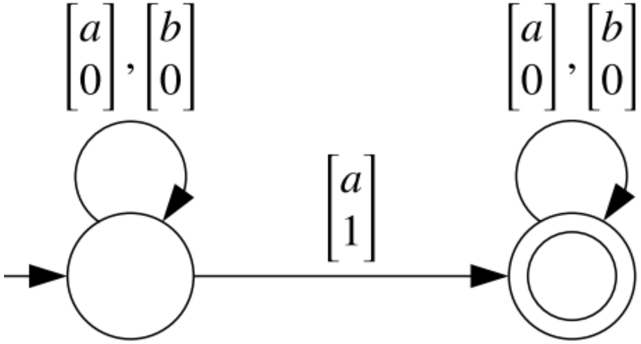


$[Q_b(x)]$

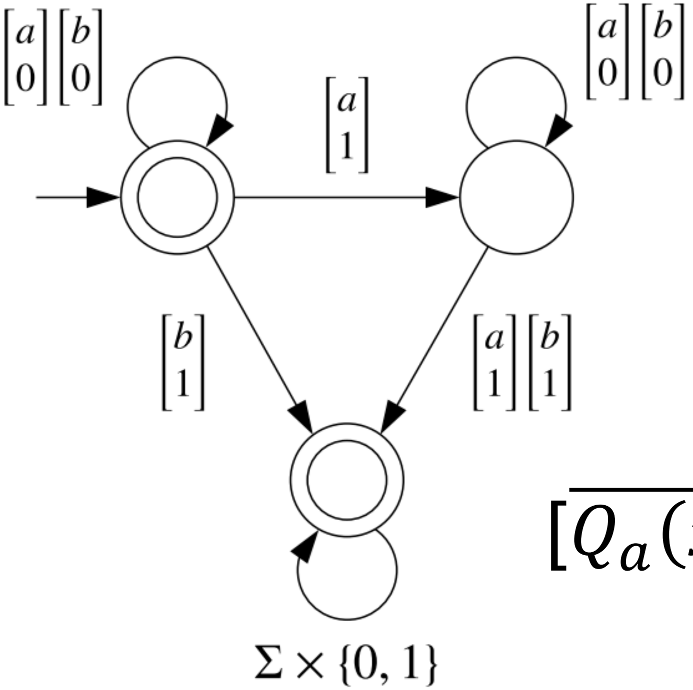


$[\exists x\,(\text{last}(x) \wedge Q_b(x))]$

$[\neg Q_a(x)]$

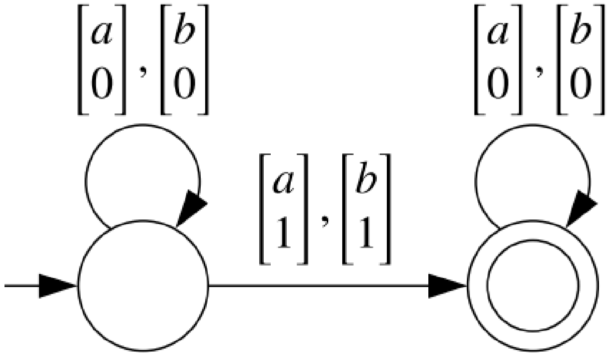


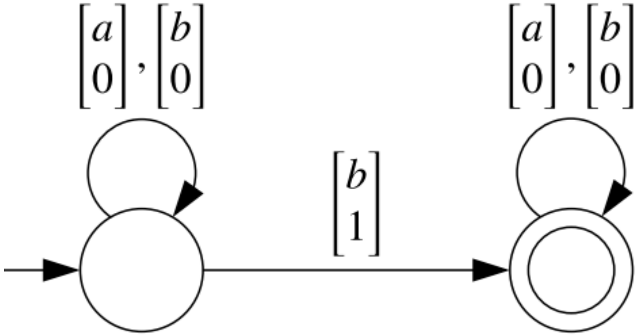
$[Q_a(x)]$



$[\overline{Q_a(x)}]$

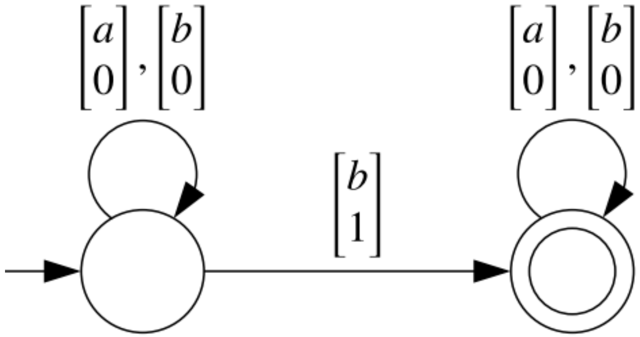
$Enc(Q_a(x))$



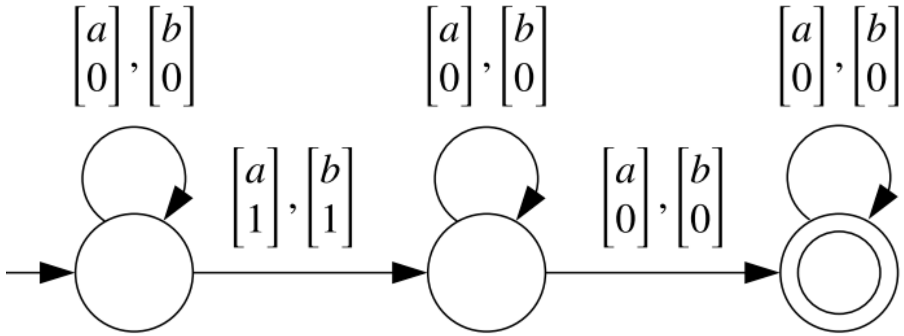


$[\neg Q_a(x)]$

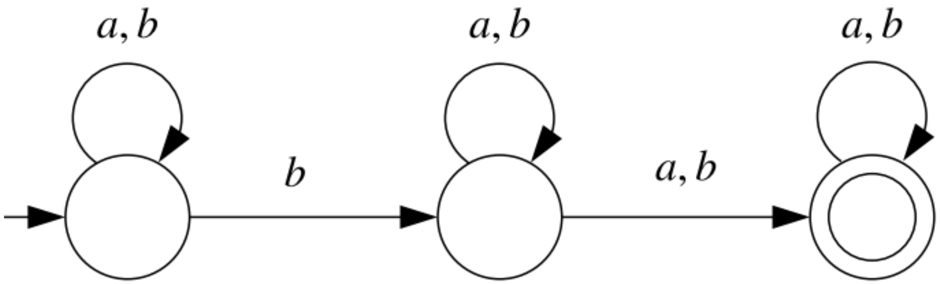
$[\neg\exists x\,(\neg \mathrm{last}(x) \wedge \neg Q_a(x))]$



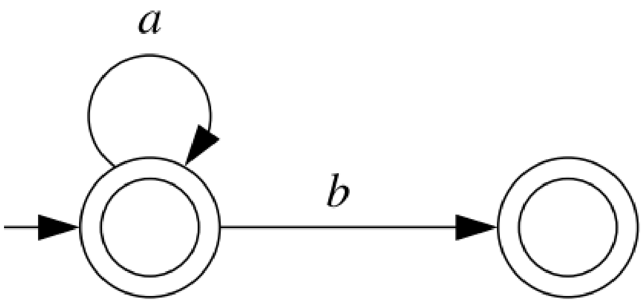
$[\neg Q_a(x)]$



$[\neg last(x)]$

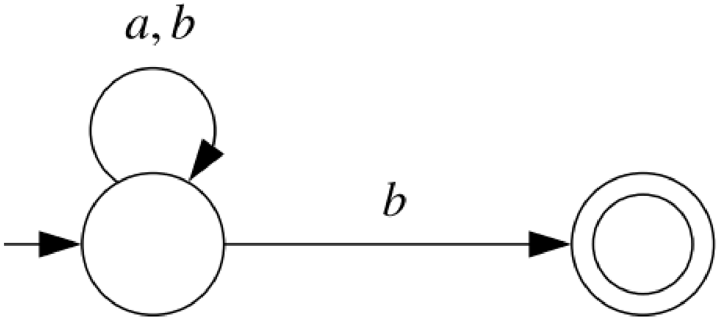


$[\exists x\,(\neg last(x) \wedge \neg Q_a(x))]$

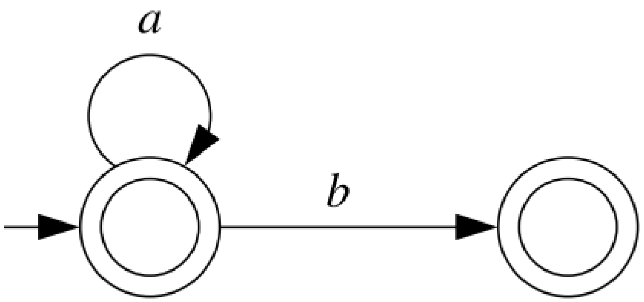


$[\neg\exists x\,(\neg last(x) \wedge \neg Q_a(x))]$

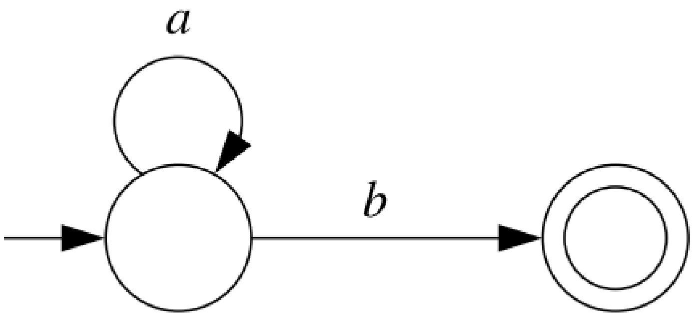
$[\exists x\ (\text{last}(x) \wedge Q_b(x)) \wedge \neg\exists x\ (\neg\text{last}(x) \wedge \neg Q_a(x))]$



$[\exists x\ (\text{last}(x) \wedge Q_b(x))]$



$[\neg\exists x\ (\neg last(x) \wedge \neg Q_a(x))]$



$[\exists x\ (\text{last}(x) \wedge Q_b(x)) \ \wedge\ \neg\exists x\ (\neg\text{last}(x) \wedge \neg Q_a(x))]$