ω-Automata

ω-Automata

- Automata that accept (or reject) words of infinite length.
- Languages of infinite words appear:
  - in verification, as encodings of non-terminating executions of a program.
  - in arithmetic, as encodings of sets of real numbers.

ω-Languages

- An ω-word is an infinite sequence of letters.
- The set of all ω-words is denoted by Σ^ω.
- An ω-language is a subset of Σ^ω.
- A language L_1 can be concatenated with an ω-language L_2 to yield the ω-language L_1L_2, but two ω-languages cannot be concatenated.
- The ω-iteration of a language $L \subseteq \Sigma^*$, denoted by L^ω, is an ω-language.
- Observe:
  - $\{ab\}^*$ contains infinitely many words, $\{ab\}^\omega$ contains only one
  - $\emptyset^\omega = \{\epsilon\}^\omega = \emptyset$

ω-Regular Expressions

- ω-regular expressions have syntax

$$s ::= r^{\omega} \mid r s_1 \mid s_1 + s_2$$

  where r is an (ordinary) regular expression.

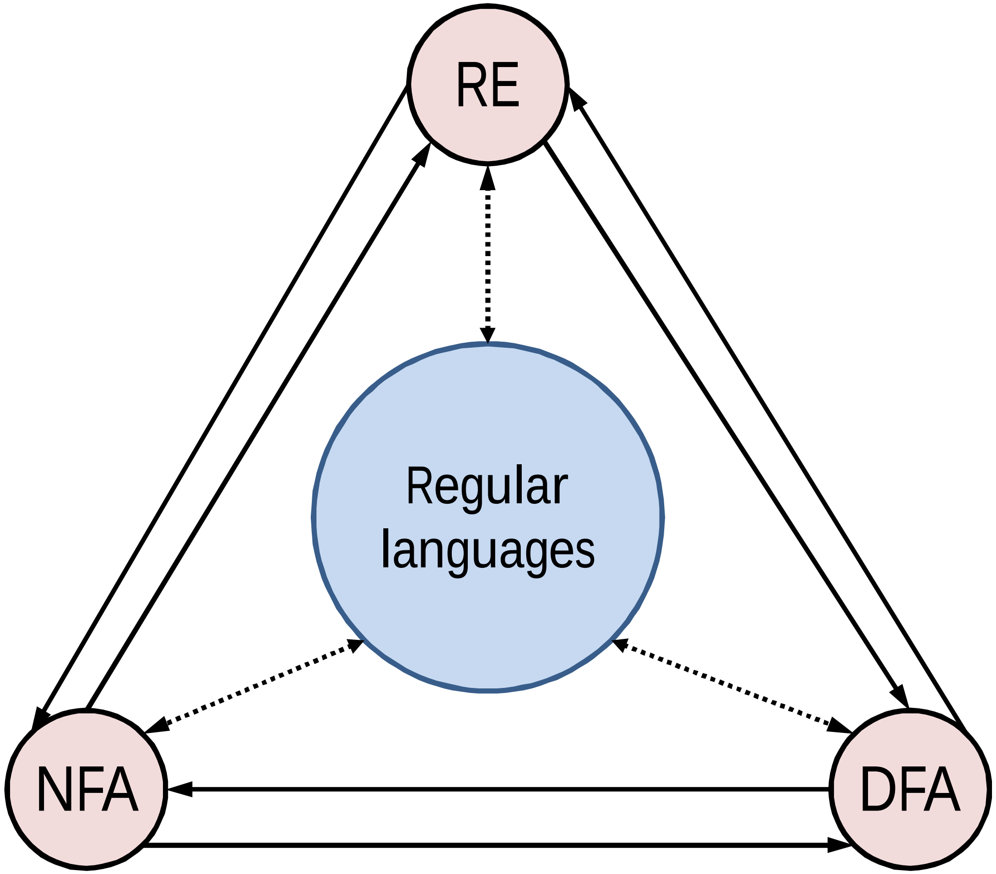
- The ω-language $L_{\omega}(s)$ of an ω-regular expression s is inductively defined by

$$L_{\omega}(r^{\omega}) = \big(L(r)\big)^{\omega} \quad L_{\omega}(r s_1) = L(r) L_{\omega}(s_1)$$

$$L_{\omega}(s_1 + s_2) = L_{\omega}(s_1) \cup L_{\omega}(s_2)$$

- An ω-language is ω-regular if it is the language of some ω-regular expression .

The Quest for a Trinity



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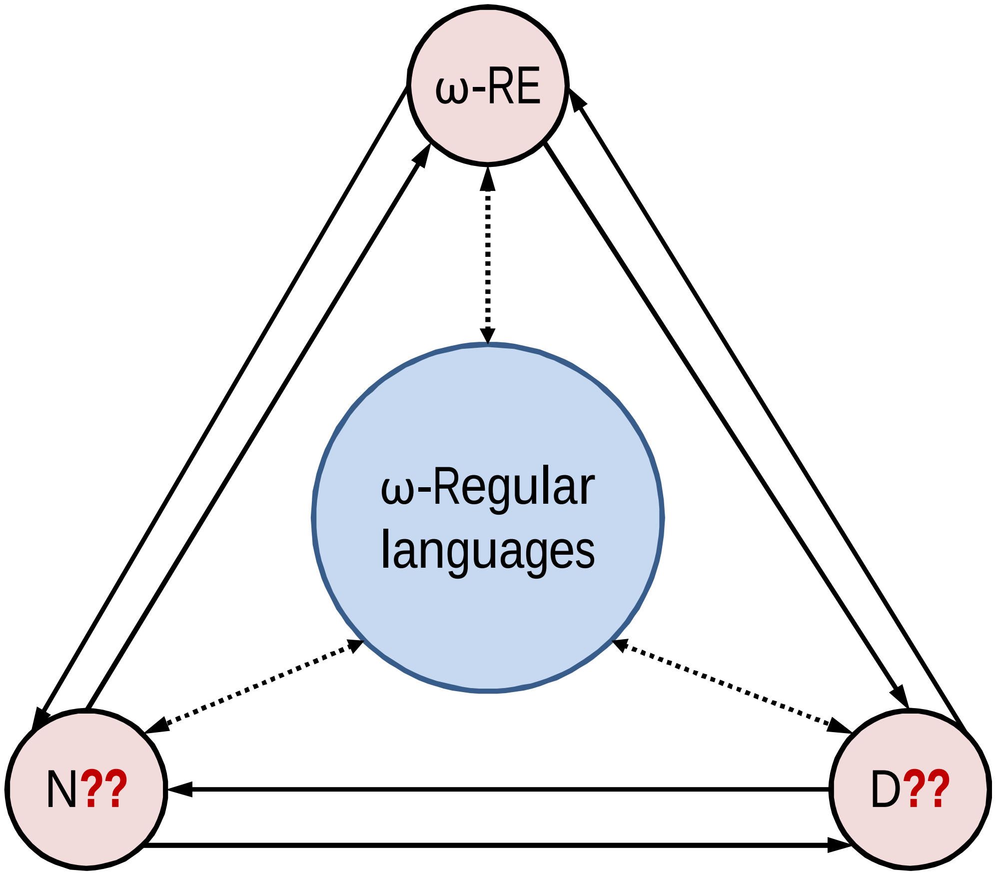

The Rules of the Quest

- Automata should still have states, transitions, and initial states, only the acceptance condition can change.

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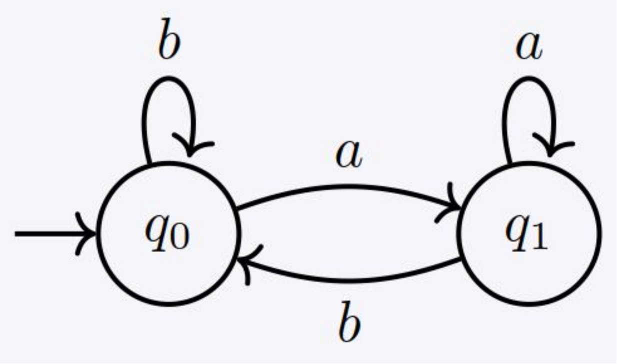
The Rules of the Quest

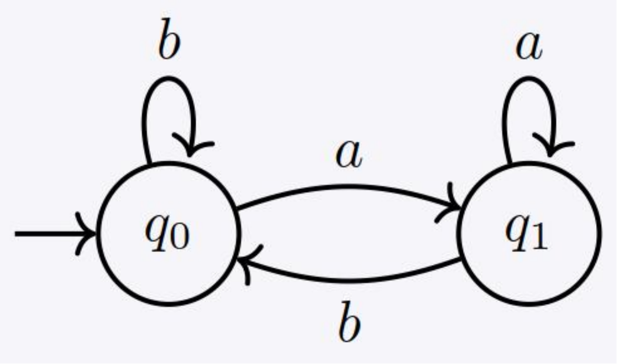
- Automata should still have states, transitions, and initial states, only the acceptance condition can change.
- For automata on finite words the acceptance condition depends only on the last state of a run (i.e., runs that end in the same state are all accepting or rejecting).
- For automata on infinite words we choose:
  the acceptance condition depends only on the
  set of states visited infinitely often by a run
  (i.e., runs that visit the same states infinitely often are all accepting or rejecting).

Basic notions: Semi-automata



- A semi-automaton is a tuple $S = (Q, \Sigma, \delta, Q_0)$ of states, alphabet, transitions, and initial states.

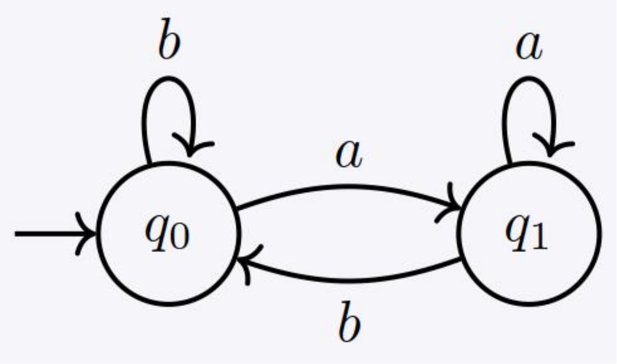
Basic notions: Runs



- A run of a semi-automaton is an infinite sequence of states and transitions starting at an initial state

  - $\rho_1 = q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \cdots$
  - $\rho_2 = q_0 \xrightarrow{b} q_0 \xrightarrow{b} q_0 \xrightarrow{b} q_0 \xrightarrow{b} q_0 \cdots$
  - $\rho_3 = q_0 \xrightarrow{a} q_1 \xrightarrow{b} q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1 \cdots$

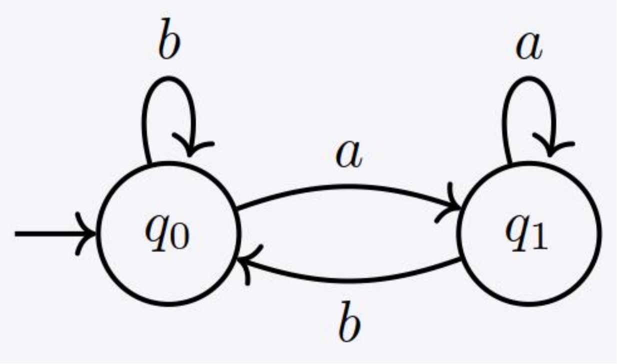
Basic notions: Runs



- The set of states visited infinitely often by a run ρ is denoted $\mathrm{inf}(\rho)$

  - $\rho_1 = q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \cdots$ $\mathrm{inf}(\rho_1) = \{q_1\}$
  - $\rho_2 = q_0 \xrightarrow{b} q_0 \xrightarrow{b} q_0 \xrightarrow{b} q_0 \xrightarrow{b} q_0 \cdots$ $\mathrm{inf}(\rho_2) = \{q_0\}$
  - $\rho_3 = q_0 \xrightarrow{a} q_1 \xrightarrow{b} q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1 \cdots$ $\mathrm{inf}(\rho_3) = \{q_0, q_1\}$

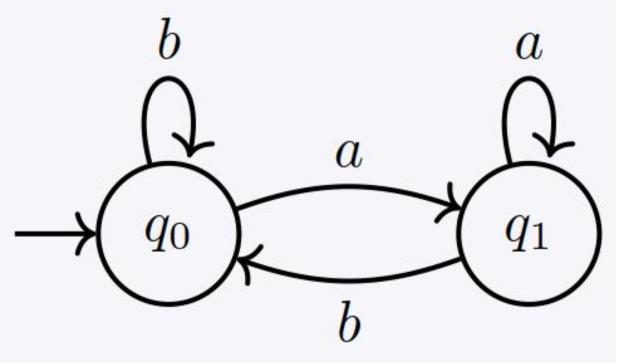
Basic notions: Acceptance conditions



- An acceptance condition is a mapping $\alpha: 2^Q \rightarrow \{0,1\}$ that determines for every set $Q' \subseteq Q$ of states whether the runs ρ with $\inf(\rho) = Q'$ are accepting or not.

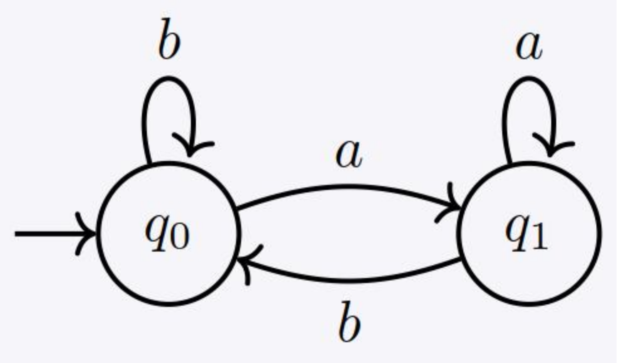
  - α_1: $\{q_0\} \mapsto 0$, $\{q_1\} \mapsto 0$, $\{q_0, q_1\} \mapsto 0$
  - α_2: $\{q_0\} \mapsto 0$, $\{q_1\} \mapsto 1$, $\{q_0, q_1\} \mapsto 1$

Basic notions: ω-Automata



- An ω-automaton is a pair $A = (S, \alpha)$, where S is a semi-automaton and α is an acceptance condition

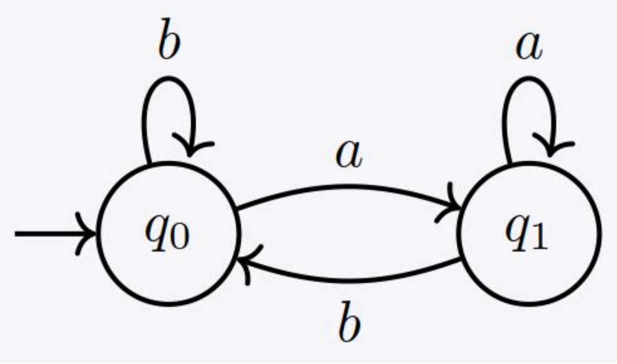
Basic notions: ω-Language



- An ω-automaton A accepts an ω-word if it has at least one accepting run on it. The ω-language $L_\omega(A)$ of A is the set of ω-words it accepts.

  - α_1: $\{q_0\} \mapsto 0, \{q_1\} \mapsto 0, \{q_0, q_1\} \mapsto 0$ $\emptyset$
  - α_2: $\{q_0\} \mapsto 0, \{q_1\} \mapsto 1, \{q_0, q_1\} \mapsto 1$?

Basic notions: ω-Language



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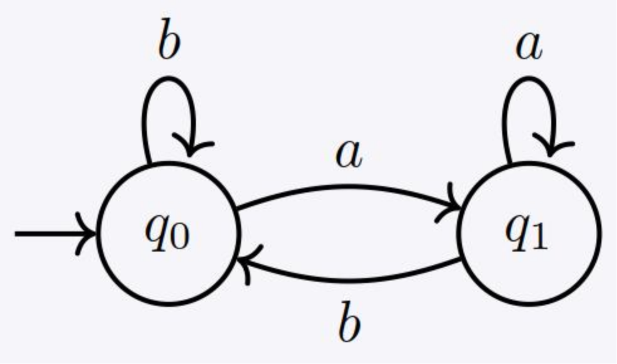
  - α_2: $\{q_0\} \mapsto 0, \{q_1\} \mapsto 1, \{q_0, q_1\} \mapsto 1$ infinitely many a

Basic notions: ω-Language



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- α_2: $\{q_0\} \mapsto 0, \{q_1\} \mapsto 1, \{q_0, q_1\} \mapsto 1$ $(b^*a)^\omega$

Types of ω-automata

- There are many different types of acceptance conditions (Büchi, co-Büchi, Rabin, Streett, parity, Muller, generalized Büchi, Emerson-Lei …)

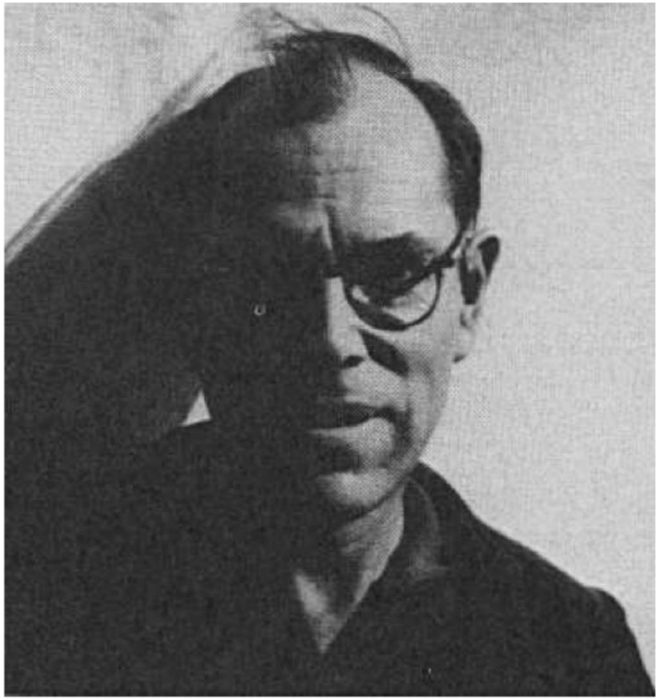
  They lead to different types of ω-automata: Büchi automata, co-Büchi automata, etc.

- A type is defined by stating a property that an acceptance condition may or may not satisfy. The type is the subset of all possible acceptance conditions that satisfy the property.

- This set of slides explains why this variety is needed.

Büchi automata

- Invented by J.R. Büchi, swiss logician.



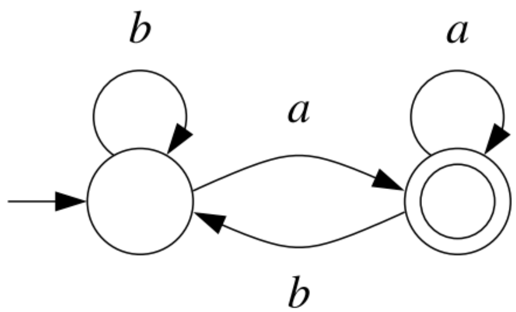
Büchi automata

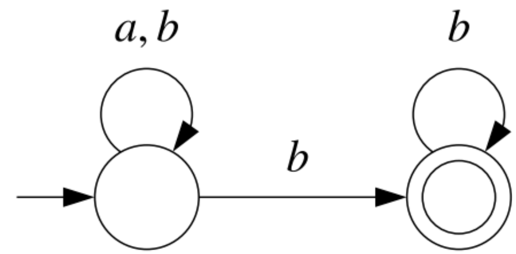
- An acceptance condition $\alpha: 2^Q \rightarrow \{0,1\}$ is a Büchi condition if there is a set $F \subseteq Q$ of accepting states such that $\alpha(Q') = 1$ iff $Q' \cap F \neq \emptyset$.
  - $\{q_0\} \mapsto 0, \{q_1\} \mapsto 1, \{q_0, q_1\} \mapsto 1$ is Büchi $F = \{q_1\}$
  - $\{q_0\} \mapsto 0, \{q_1\} \mapsto 1, \{q_0, q_1\} \mapsto 0$ is not Büchi

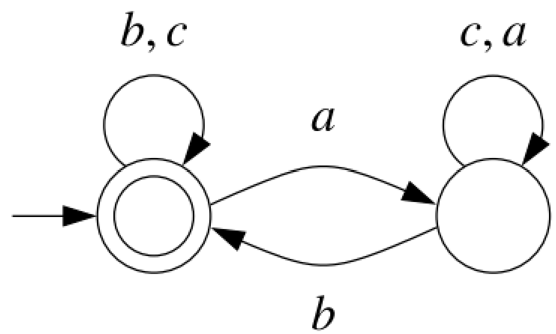
- By definition, a run ρ is accepting iff $\inf(\rho) \cap F \neq \emptyset$ iff (in words) ρ visits F infinitely often.

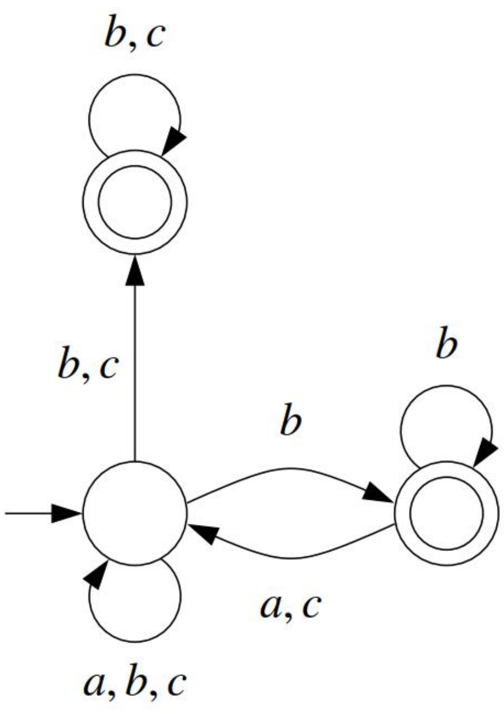
- A Büchi condition α is completely determined by F. We write $A = (S, F) = (Q, \Sigma, \delta, Q_0, F)$.

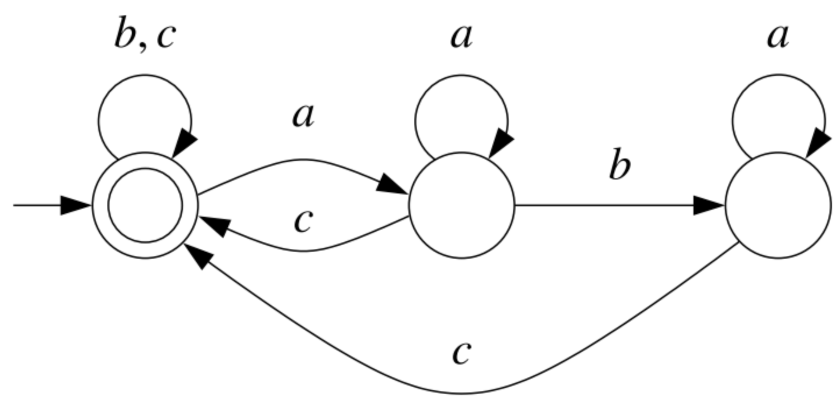
Some Büchi automata



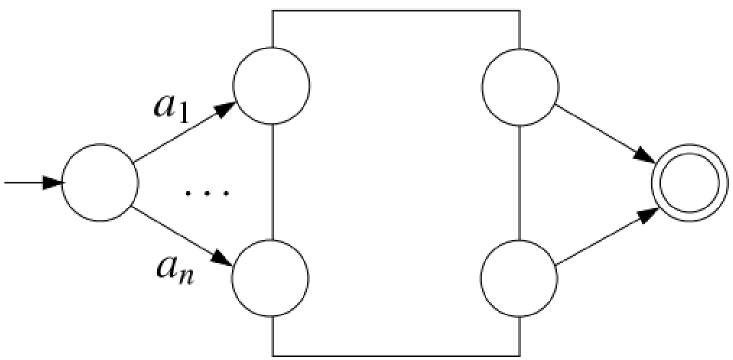






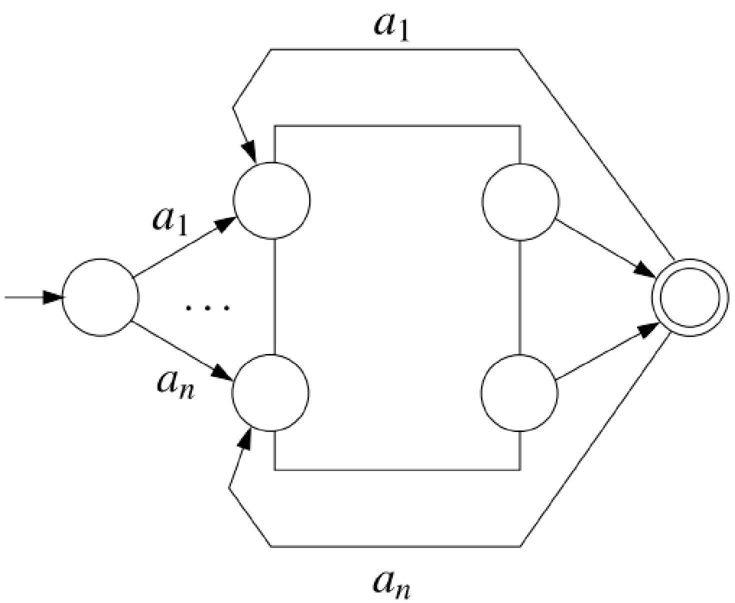


From ω-regular expressions to NBAs



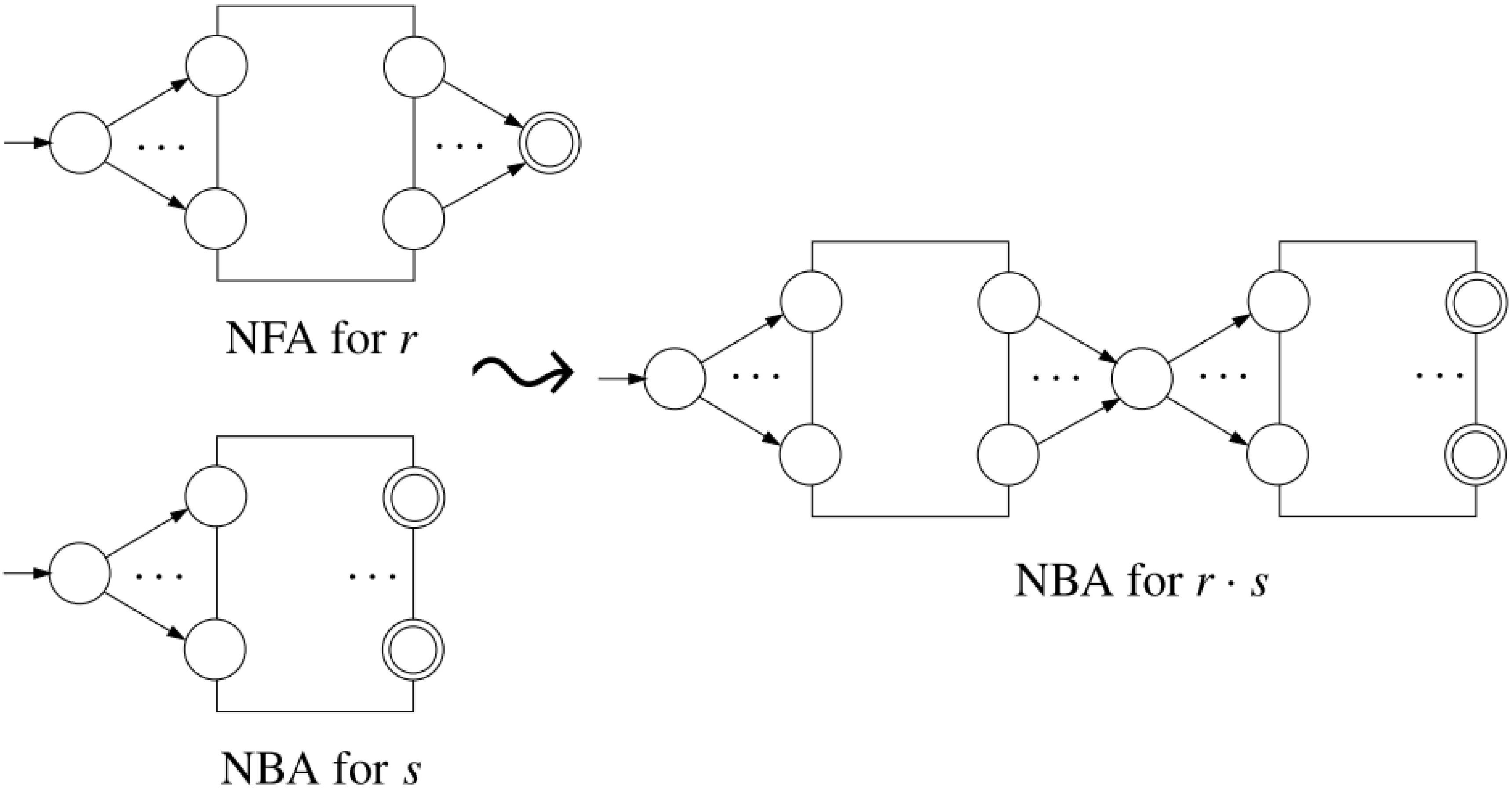
NFA for r

$\rightsquigarrow$



NBA for r^ω

From ω-regular expressions to NBAs



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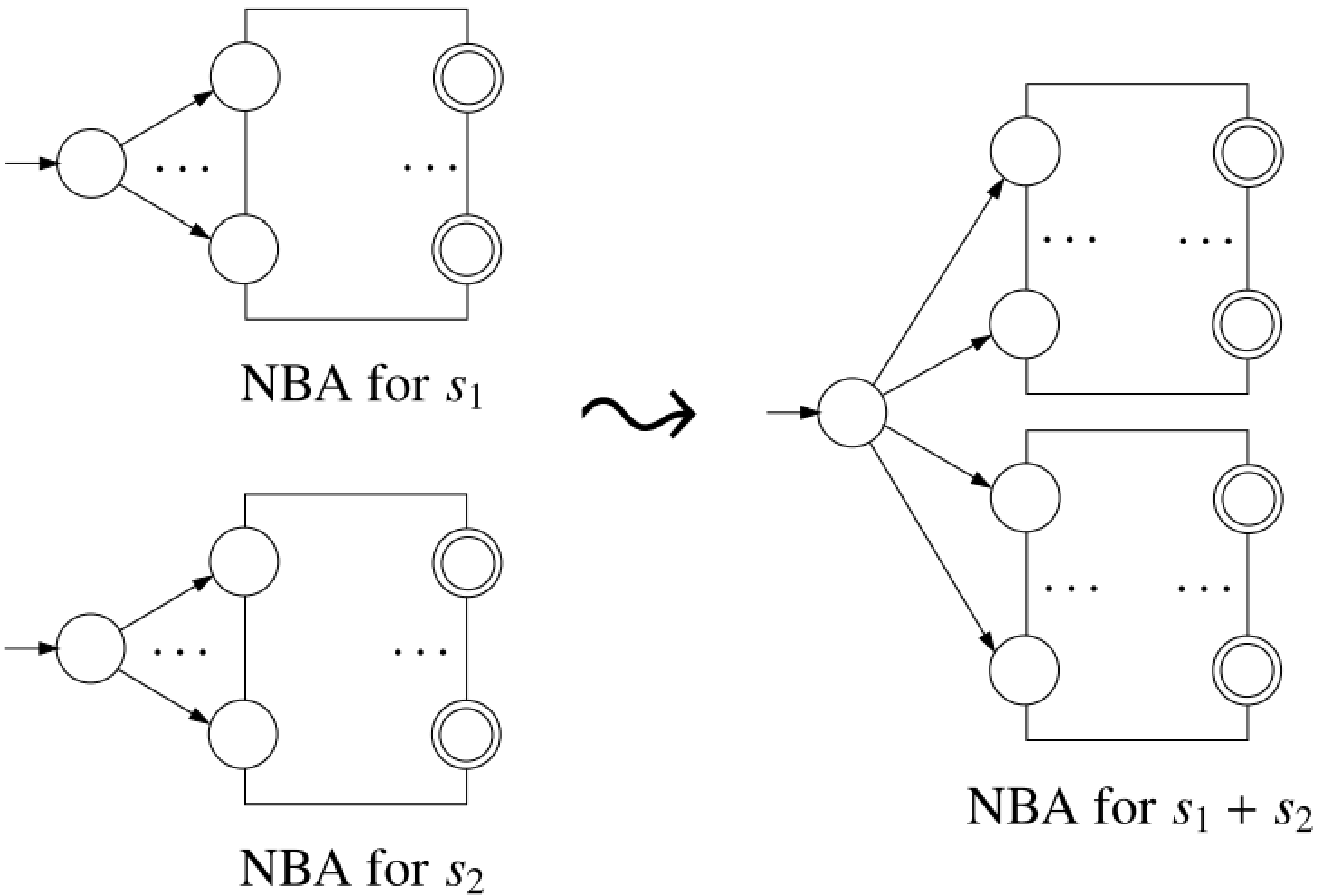
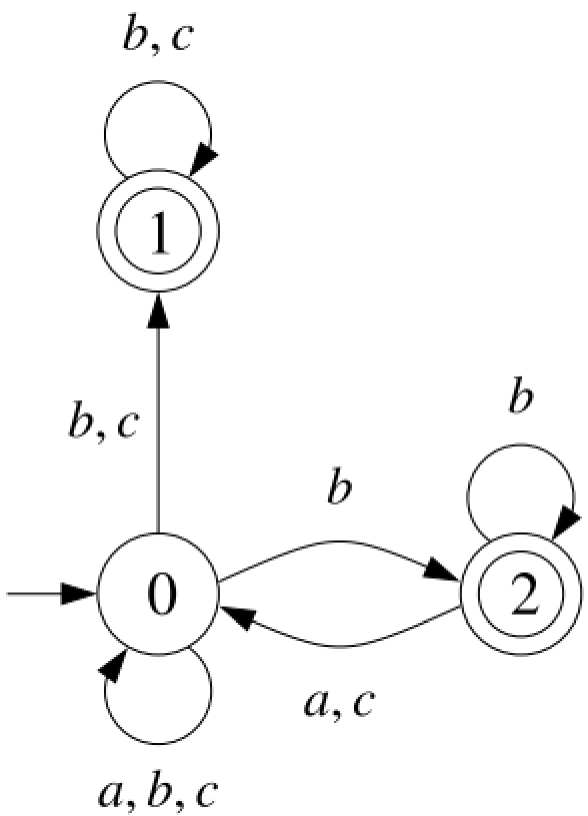


From NBAs to ω-regular expressions

- Lemma: Let A be a NFA, and let q, q' be states of A. The language $L_q^{q'}$ of words with runs leading from q to q' and visiting q' exactly once after leaving q is regular.
- Let $r_q^{q'}$ denote a regular expression for $L_q^{q'}$.

From NBAs to ω-regular expressions

- Example:



$$
\begin{aligned}
r_0^1 &= (a+b+c)^*(b+c) \\
r_0^2 &= (a+b+c)^*b \\
r_1^1 &= (b+c) \\
r_2^2 &= b+(a+c)(a+b+c)^*b
\end{aligned}
$$

From NBAs to ω-regular expressions

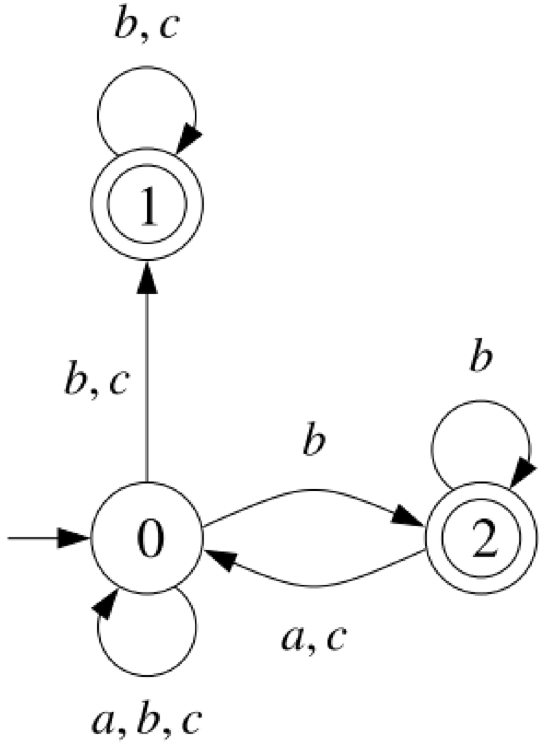
- Given a NBA A , we look at it as a NFA, and compute regular expressions $r_q^{q'}$.
- We show:

$$L_\omega(A) = L\left(\sum_{q\in F} r_{q_0}^{q} \left(r_q^q\right)^\omega\right)$$

  – An ω-word belongs to $L_\omega(A)$ iff it is accepted by a run that starts at q_0 and visits some accepting state q infinitely often.

From NBAs to ω-regular expressions

- Example:



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\end{aligned}
$$

$$L_\omega(A) = r_0^1\,(r_1^1)^\omega + r_0^2\,(r_2^2)^\omega$$

DBAs are less expressive than NBAs

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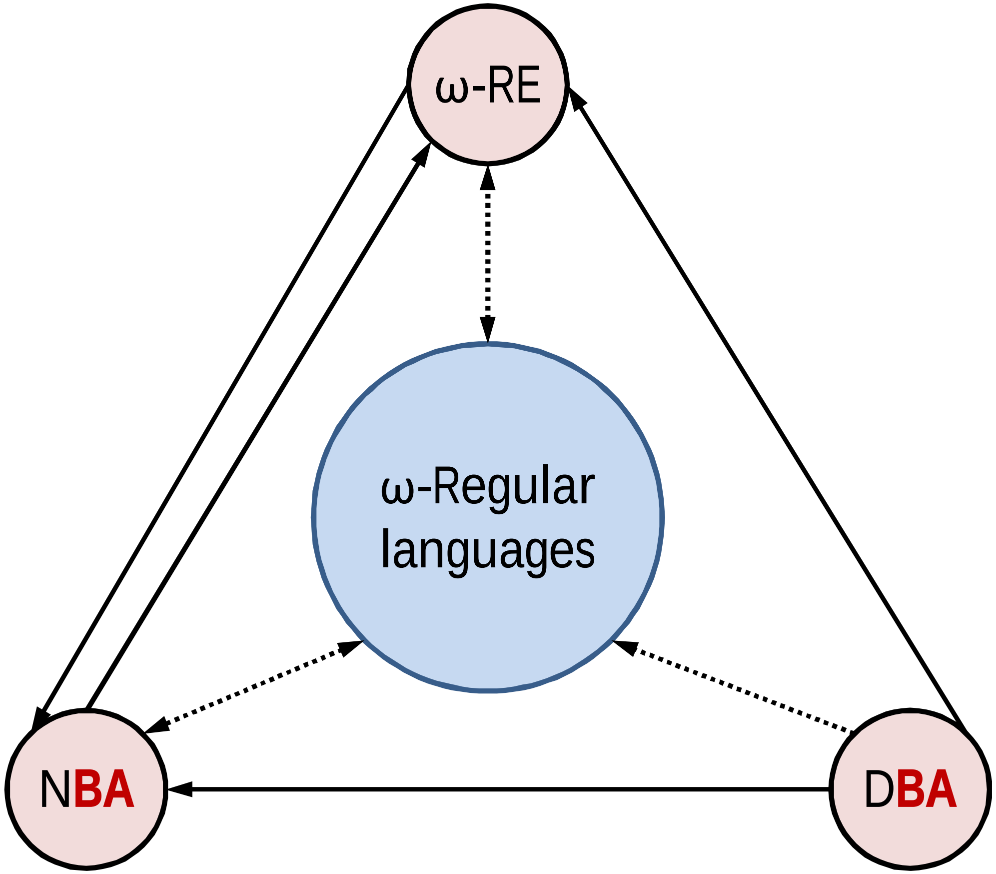
    DBA accepts $b^{n_0} a\, b^{n_1}\, a b^\omega$ → DFA accepts $b^{n_0} a\, b^{n_1} a\, b^{n_2}$ etc.
  - By determinism and finite number of states, the DBA accepts

$$b^{n_0} a\, b^{n_1} a\, b^{n_2} \ldots a\, b^{n_i} (a b^{n_{i+1}} \ldots a b^{n_j})^\omega$$

for some $i < j$. This word does not belong to $(a+b)^* b^\omega$.

Büchi automata do not form a Trinity



Co-Büchi automata

- An accepting condition $\alpha: 2^Q \rightarrow \{0,1\}$ is a co-Büchi condition if there is a set F of accepting states such that $\alpha(Q') = 1$ iff $Q' \cap F = \emptyset$.
  - $\{q_0\} \mapsto 0, \{q_1\} \mapsto 1, \{q_0, q_1\} \mapsto 0$ is co-Büchi $F = \{q_0\}$
  - $\{q_0\} \mapsto 0, \{q_1\} \mapsto 1, \{q_0, q_1\} \mapsto 1$ is not co-Büchi

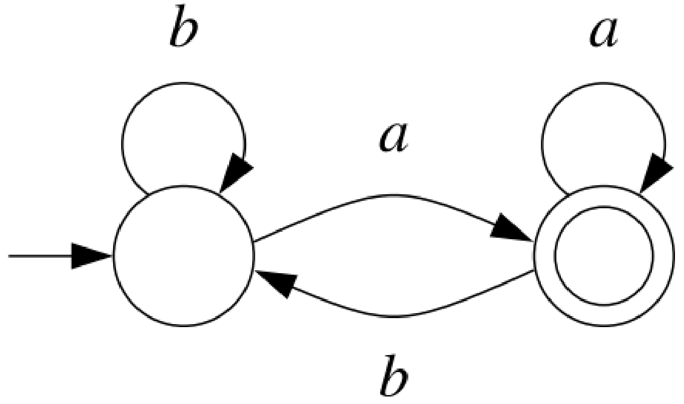
- Equivalently: ρ is accepting iff $\inf(\rho) \cap F = \emptyset$ iff (in words) ρ visits F finitely often.

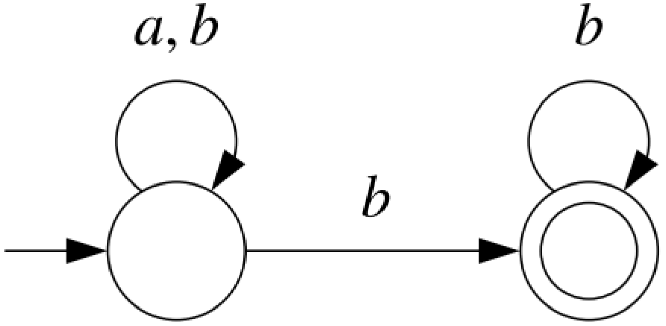
- A co-Büchi condition α is completely determined by F. We write $A = (S, F) = (Q, \Sigma, \delta, Q_0, F)$ (danger!)

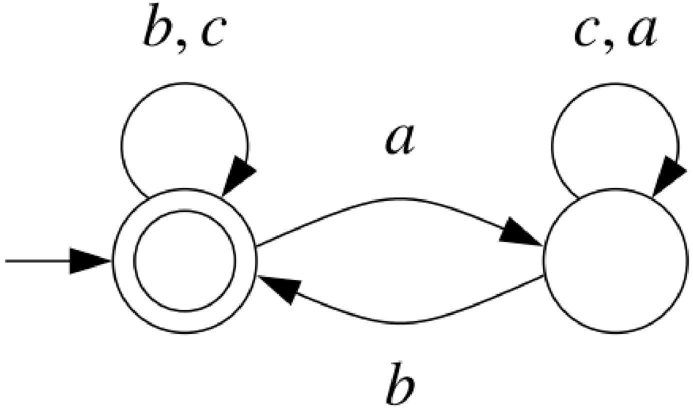
Co-Büchi automata

- Let A be a Büchi automaton, let B be the <u>same</u> co-Büchi automaton (with the same set F), and let ρ be a run:
  - ρ is accepting in A if it visits F infinitely often
  - ρ is accepting in B if it visits F finitely often
- So an accepting run of A is a rejecting run of B and vice versa.
- Therefore: If A is a **D**BA recognizing an ω-language L, then B is a **D**CA recognizing $\overline{L}$.
- Not necessarily true for NBA !

Which are the languages?



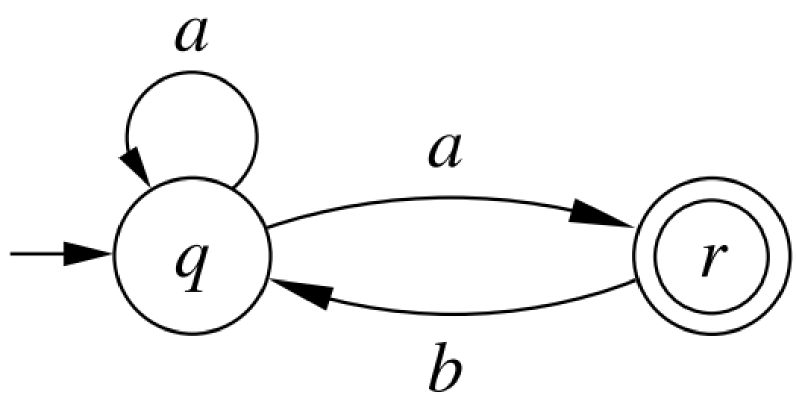


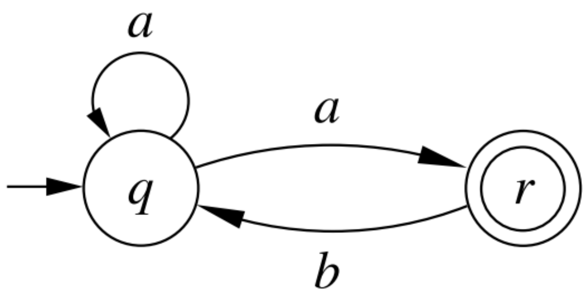


Determinizing co-Büchi automata

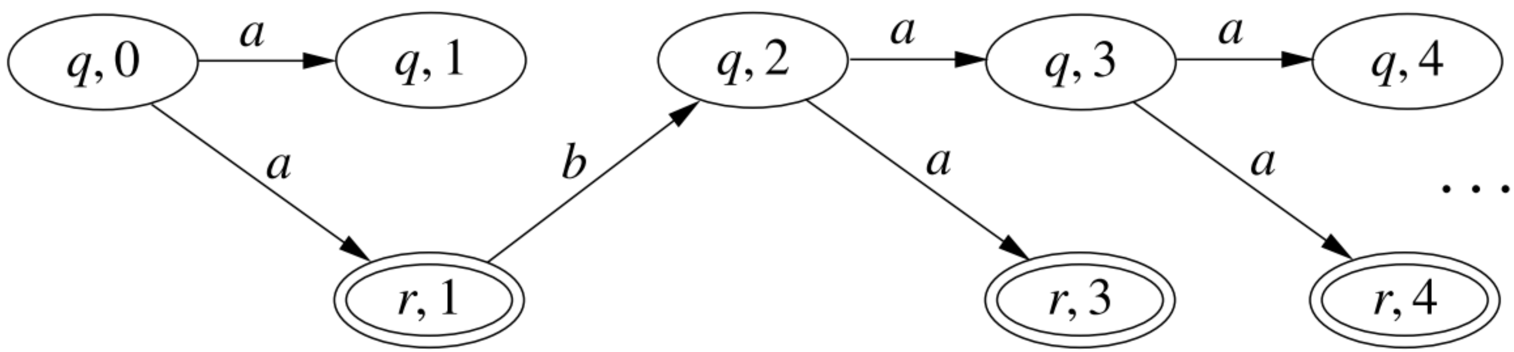
- Given a NCA A we construct a DCA B such that $L(A) = L(B)$.
- We proceed in three steps:
  - We assign to every ω-word w a directed acyclic graph $dag(w)$ that ``contains´´ all runs of A on w.
  - We prove that w is accepted by A iff $dag(w)$ is infinite but contains only finitely many breakpoints.
  - We construct a DCA B such that w is accepted by B iff $dag(w)$ is infinite but contains only finitely many breakpoints.

- Running example:





$dag(aba^{\omega})$



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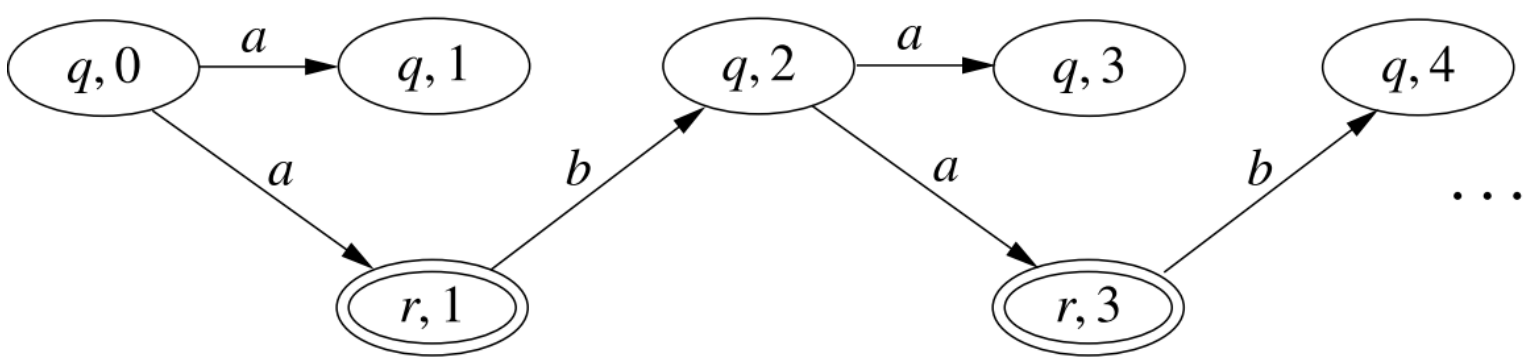
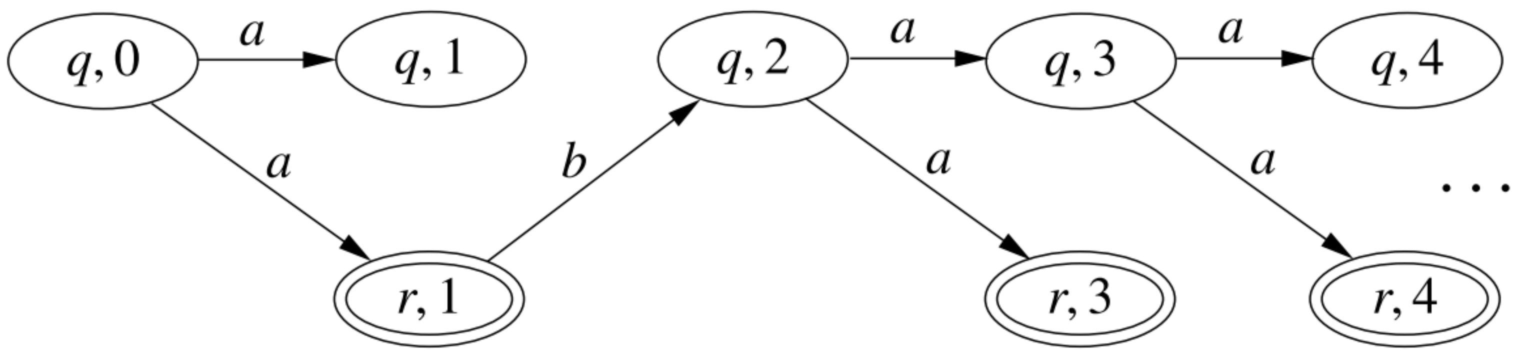
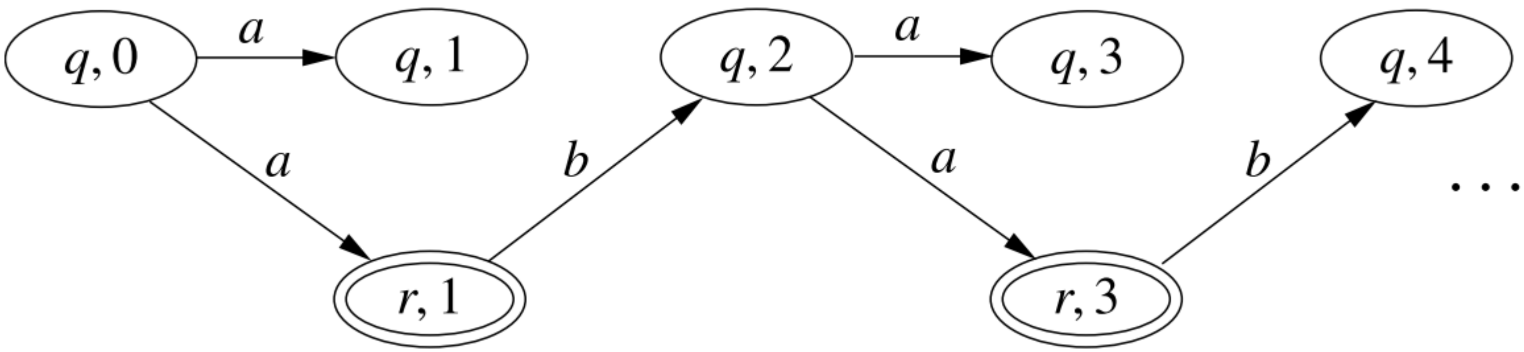
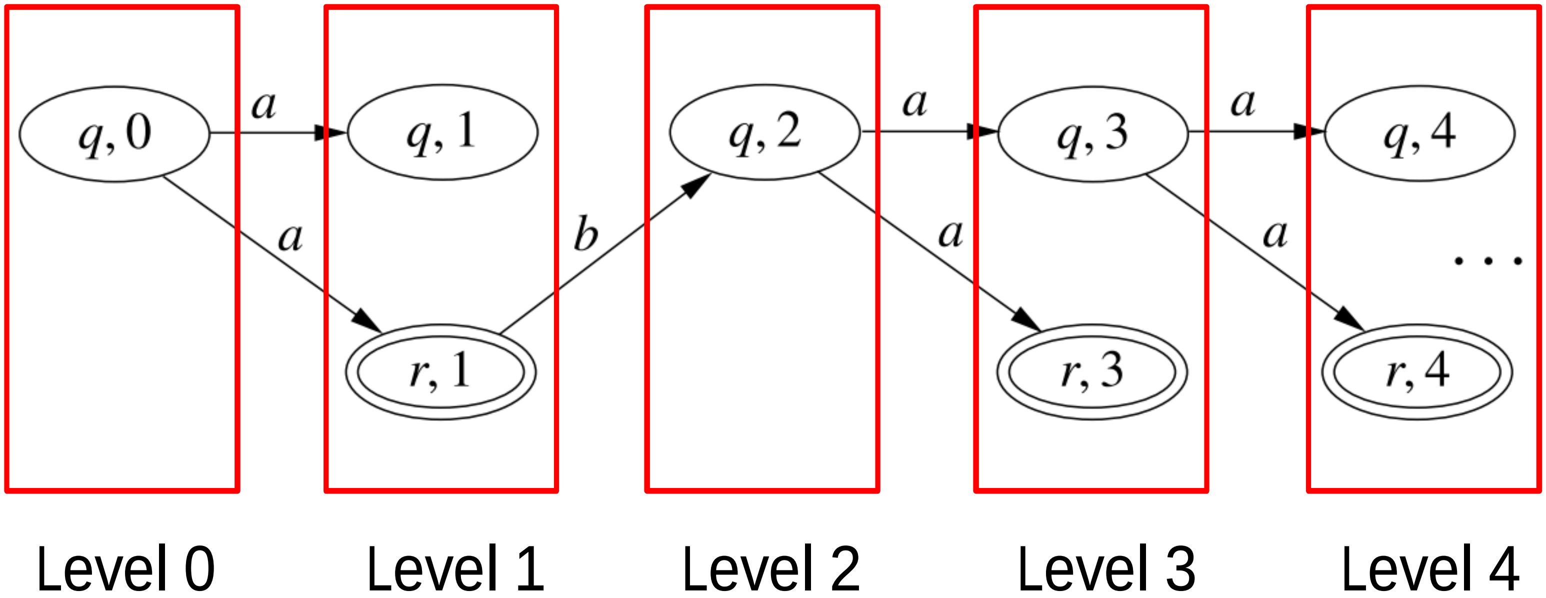


- A accepts w iff some infinite path of $dag(w)$ only visits accepting states finitely often





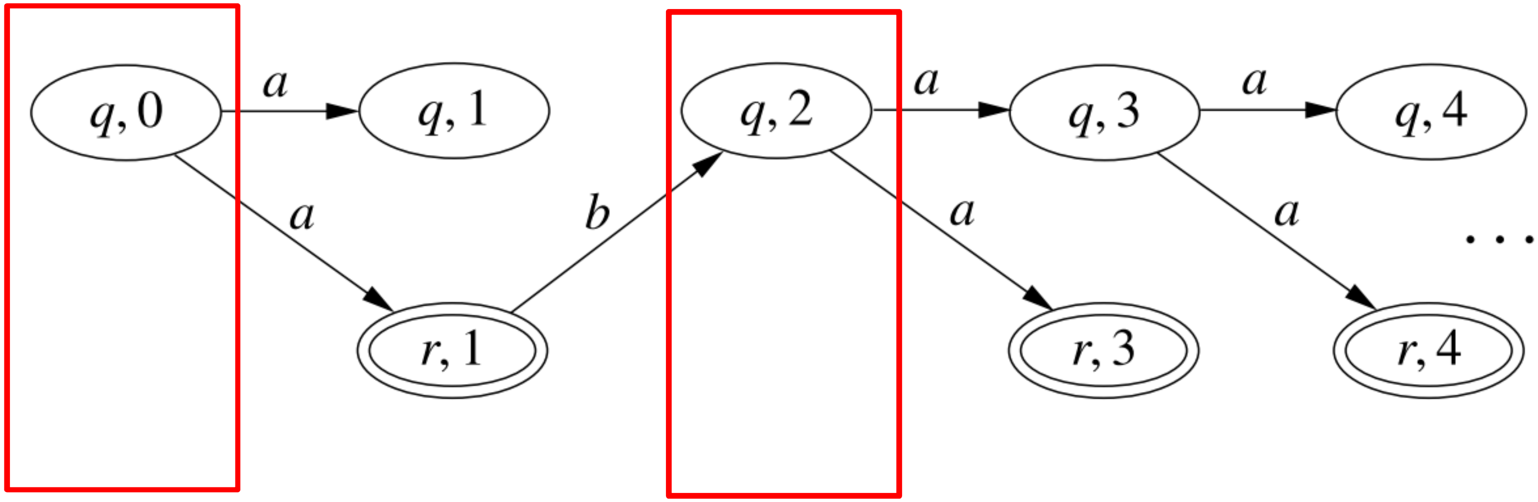
Levels of a dag

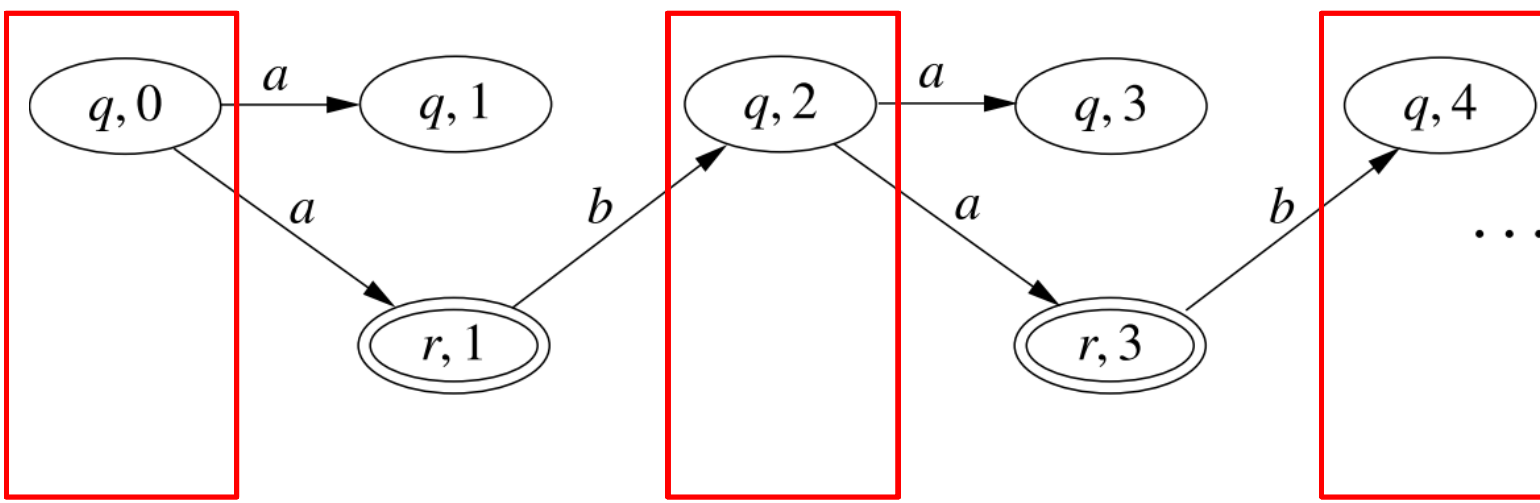


Breakpoints of a dag

- We define inductively the set of levels that are breakpoints:
  - Level 0 is always a breakpoint
  - If level l is a breakpoint, then the next level l' such that every path from l to l' visits an accepting state at some level between $l+1$ and l' is also a breakpoint.

Only two breakpoints





Infinitely many breakpoints

Lemma: A accepts w iff $dag(w)$ is infinite and has only finitely many breakpoints.

Proof:

($\Rightarrow$) If A accepts w, then it has at least one run on w, and so $dag(w)$ is infinite.

Moreover, the run visits accepting states only finitely often, and so after it stops visiting accepting states there are no further breakpoints.

Lemma: A accepts w iff $dag(w)$ is infinite and has only finitely many breakpoints.

Proof:

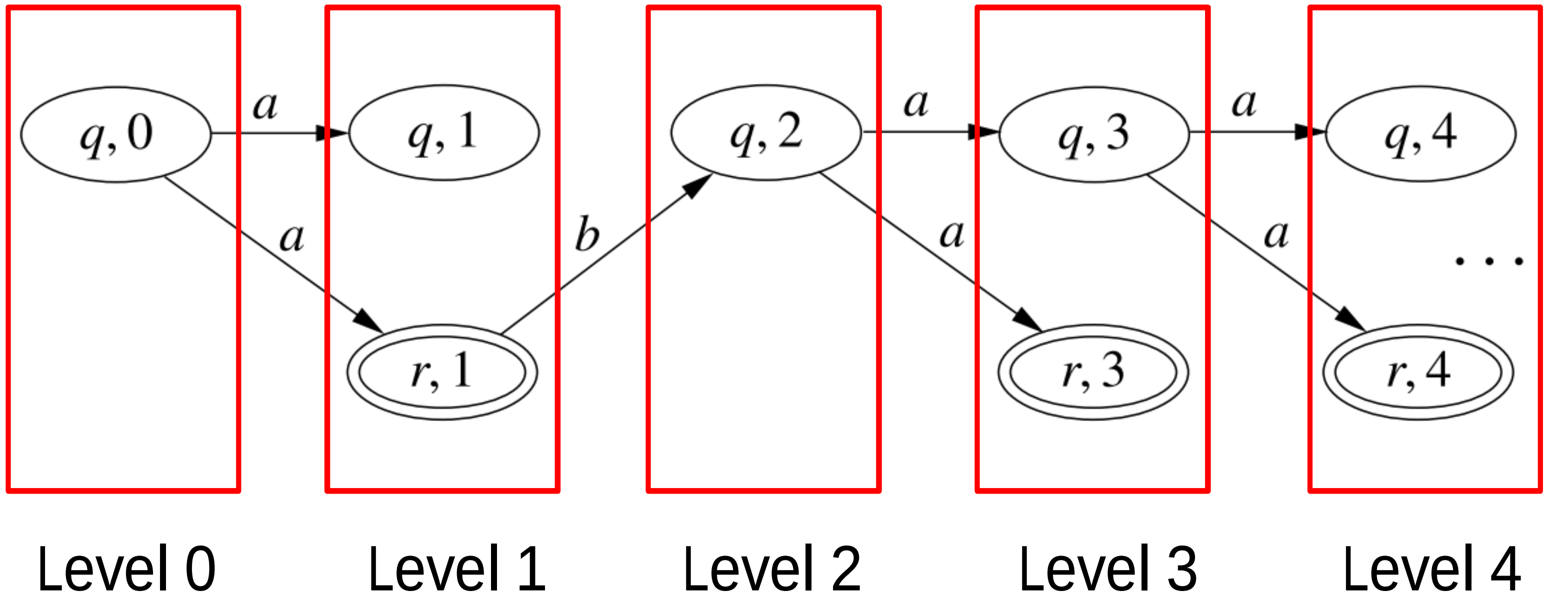
($\Leftarrow$) Assume $dag(w)$ is infinite and has only finitely many breakpoints. Let l be the last breakpoint.

Since $dag(w)$ is infinite, for every $l' > l$ there is a path from l to l' that visits no accepting states.

The subdag containing all these paths is infinite and has finite degree.

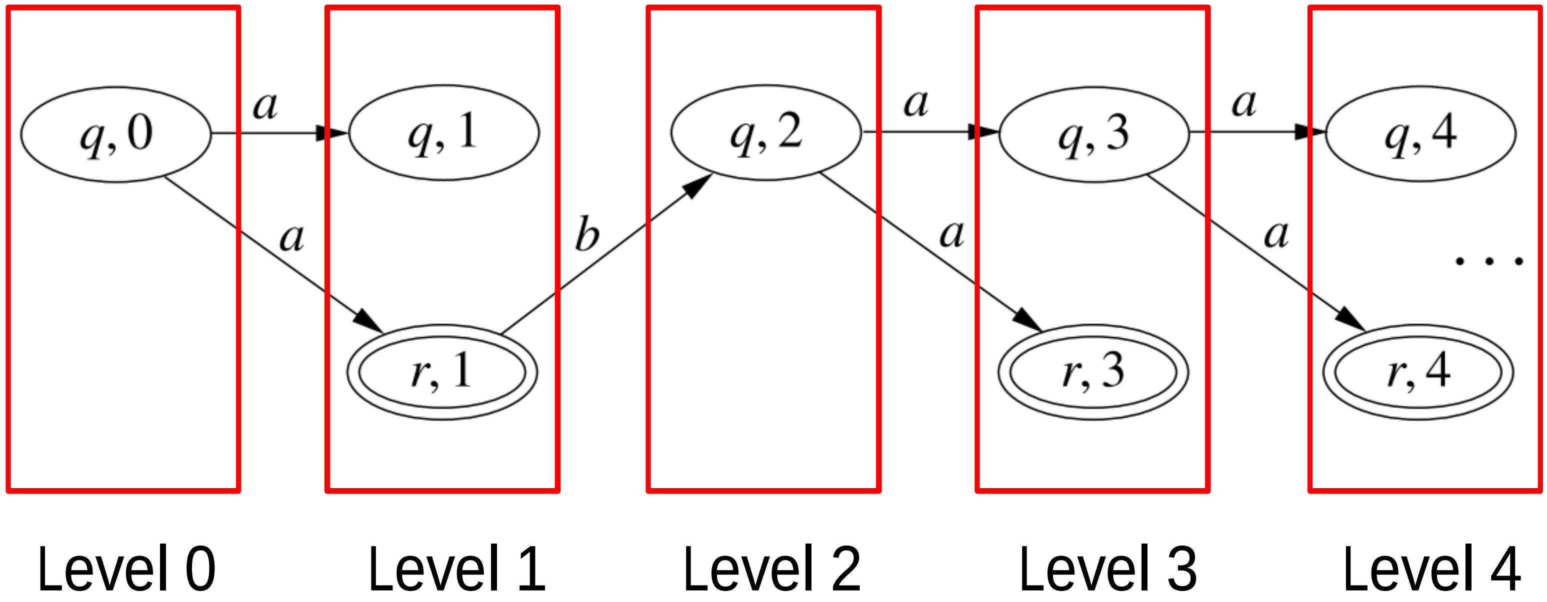
By König‘s Lemma the dag contains an infinite path.

Constructing the DCA



If we could tell if a level is a breakpoint by looking at it <u>and to no other level</u>, then we could take the set of all levels/ breakpoints as the set of states/accepting states of the DCA.

Constructing the DCA



However, in oder to decide if a level is a breakpoint we need information about its ``history´´.

Solution: add that information to the level.

Constructing the DCA

- States: pairs $[P, O]$ where:
  - P is the set of states of a level, and
  - $O \subseteq P$ is the set of states ``that owe a visit to the set of accepting states".
- Formally: $q \in O$ if q is the endpoint of a path starting at the last breakpoint that has not yet visited any accepting state.

Constructing the DCA

- **States**: pairs $[P, O]$
- **Initial state**: pair $[Q_0, \emptyset]$.
- **Transitions**: $\delta([P, O], a) = [P', O']$ where $P' = \delta(P, a)$ and O' is given by:
  - $O' = \delta(O, a) \setminus F$ if $O \neq \emptyset$

    (automaton updates set of owing states)
  - $O' = \delta(P, a) \setminus F$ if $O = \emptyset$

    (automaton starts search for next breakpoint)
- **Accepting states**: pairs $[P, \emptyset]$ (no owing states)

NCAtoDCA(A)

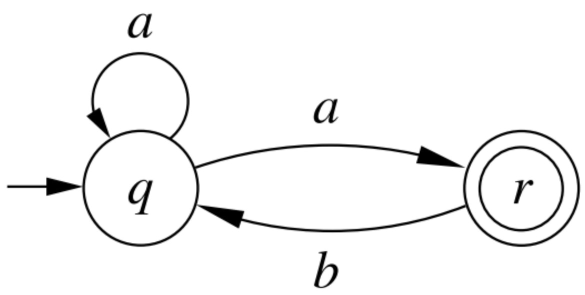
Input: NCA $A = (Q, \Sigma, \delta, Q_0, F)$

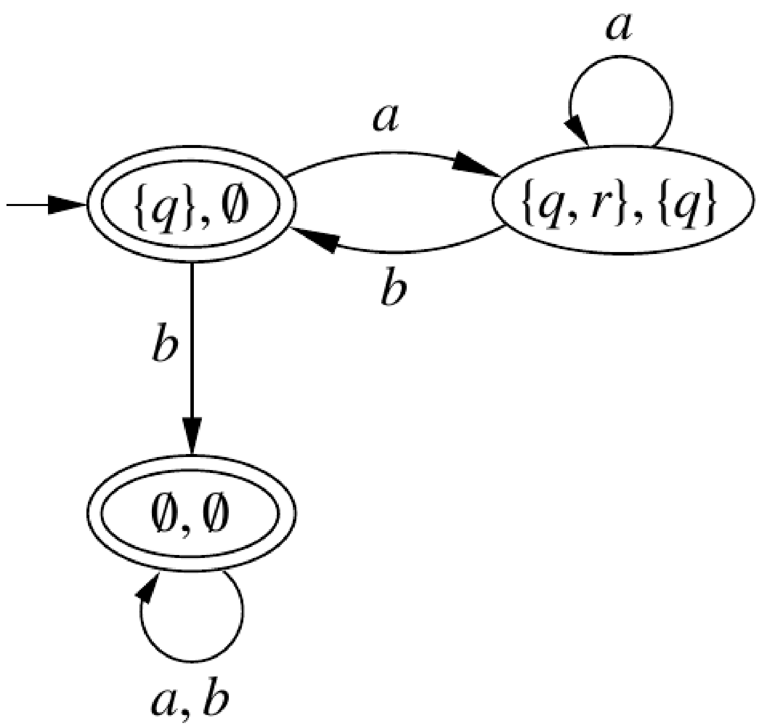
Output: DCA $B = (\tilde{Q}, \Sigma, \tilde{\delta}, \tilde{q}_0, \tilde{F})$ with $L_\omega(A) = L_\omega(B)$

$\tilde{Q}, \tilde{\delta}, \tilde{F} \leftarrow \emptyset$; $\tilde{q}_0 \leftarrow [Q_0, \emptyset]$
$W \leftarrow \{\ \tilde{q}_0\ \}$
while $W \neq \emptyset$ **do**
    pick $[P, O]$ **from** W; **add** $[P, O]$ **to** $\tilde{Q}$
    if $O = \emptyset$ **then add** $[P, O]$ **to** $\tilde{F}$
    for all $a \in \Sigma$ **do**
        $P' = \delta(P, a)$
        if $O \neq \emptyset$ **then** $O' \leftarrow \delta(O, a) \setminus F$ **else** $O' \leftarrow \delta(P, a) \setminus F$
        add $([P, O], a, [P', O'])$ **to** $\tilde{\delta}$
        if $[P', O'] \notin \tilde{Q}$ **then add** $[P', Q']$ **to** W

- Complexity: at most 3^n states

Running example





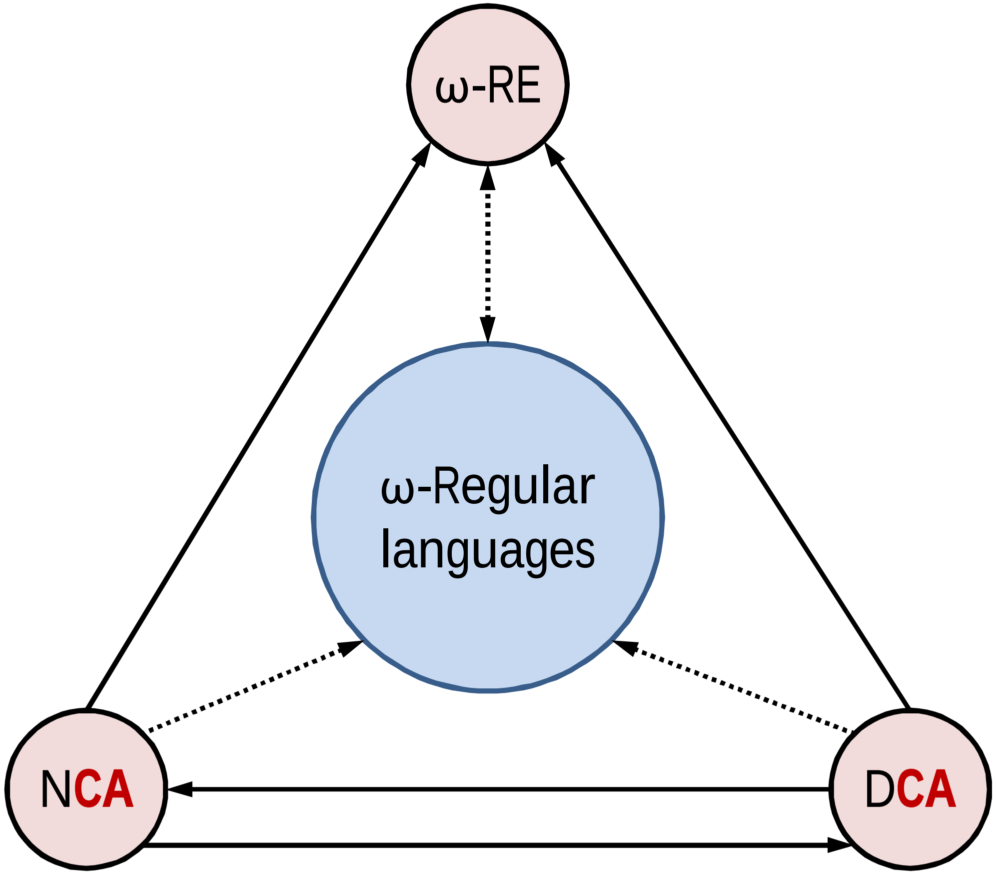
Co-Büchi Automata do not form a Trinity

Lemma: No DCA (and so no NCA) recognizes the ω-language $(b^*a)^\omega$.

Proof: Assume the contrary. Then the same automaton seen as a DBA recognizes the complement ω-language $(a+b)^*b^\omega$. Contradiction.

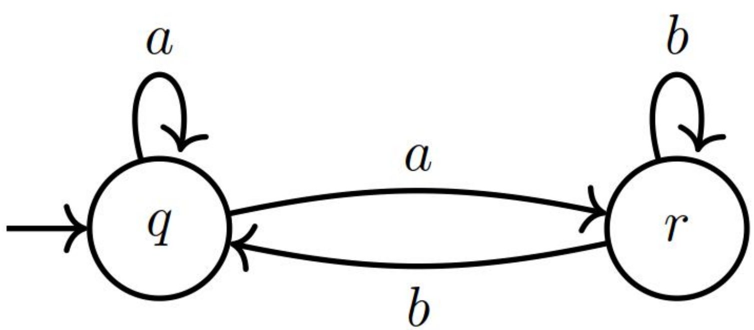
It can be proven that all ω-languages accepted by NCA are ω-regular (exercise!).
So NCA are strictly less expressive than NBA.

Co-Büchi Automata do not form a Trinity



Generalizing NBAs

- Recall: No DBA for $(a + b)^* b^\omega$
- Can be „repaired“ by combining Büchi and co-Büchi conditions:



Runs that visit q finitely often and moreover visit r infinitely often recognize $(a + b)^* b^\omega$

Rabin automata

- A Rabin pair is a pair $\langle F, G \rangle$ of sets of states.
- An accepting condition $\alpha: 2^Q \to \{0,1\}$ is a Rabin condition if there is a set $\mathcal{R}$ of Rabin pairs such that

$$\alpha(Q') = 1 \quad \text{iff} \quad Q' \cap F \neq \emptyset \text{ and } Q' \cap G = \emptyset$$

for some pair $\langle F, G \rangle \in \mathcal{R}$.

ρ is accepting

iff $\inf(\rho) \cap F \neq \emptyset$ and $\inf(\rho) \cap G = \emptyset$ for some $\langle F, G \rangle \in \mathcal{R}$

iff (in words) ρ visits F infinitely often and G finitely often

for some $\langle F, G \rangle \in \mathcal{R}$.

Rabin automata

- The accepting condition

$$\{q_0\} \mapsto 1, \{q_1\} \mapsto 1, \{q_0, q_1\} \mapsto 0$$

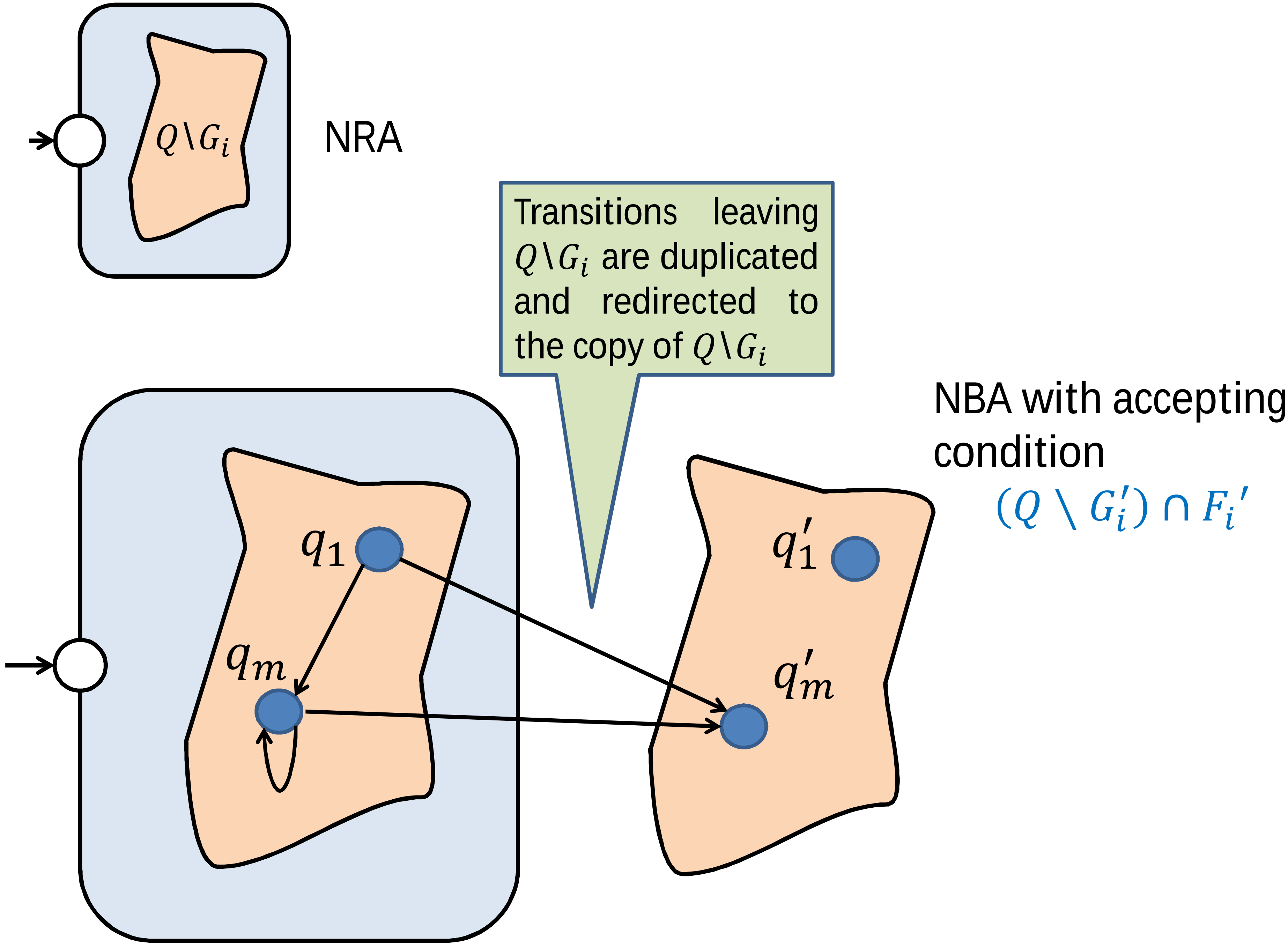
is neither Büchi nor co-Büchi, but it is the Rabin condition $\{\ \langle\{q_0\},\{q_1\}\rangle\ ,\ \langle\{q_1\},\{q_0\}\rangle\ \}$ (two Rabin pairs)

- Büchi condition F $\equiv$ Rabin condition $\{\langle F, \emptyset\rangle\}$
- Co-Büchi condition G $\equiv$ Rabin condition $\{\langle Q, G\rangle\}$

- Theorem (Safra): Any NRA with n states can be effectively transformed into a DRA with $n^{O(n)}$ states.

From Rabin to Büchi automata

- Let A be a NRA with condition $\{\langle F_1, G_1\rangle, \ldots, \langle F_m, G_m\rangle\}$.
- Let $A_1, \ldots, A_m$ be NRAs with the same semi-automaton as A but Rabin conditions $\{\langle F_1, G_1\rangle\}$, … , $\{\langle F_m, G_m\rangle\}$ respectively.
- We have: $L(A) = L(A_1) \cup \cdots \cup L(A_m)$
- We proceed in two steps:
  1. we construct for each NRA A_i an NBA A_i' such that $L(A_i) = L(A_i')$
  2. we (easily) construct an NBA A' such that
     $$L(A') = L({A'}_1) \cup \ldots \cup L({A'}_m)$$



$Q \backslash G_i$
NRA
Transitions leaving $Q \backslash G_i$ are duplicated and redirected to the copy of $Q \backslash G_i$
NBA with accepting condition
$(Q \setminus G_i') \cap F_i{}'$
q_1
q_m
q_1'
q_m'

Beyond Trinities

- Can we find a class X of ω-automata such that
  - RE, NXA, DXA form a Trinity, and
  - Boolean operations for DXAs can be implemented „as for DFAs“ ?

1) For every DXA $A = (S, \alpha)$ there is a DXA $\overline{A} = (S, \overline{\alpha})$ recognizing $\overline{L_\omega(A)}$
2) For every two DXAs $A_1 = (S_1, \alpha_1)$ and $A_2 = (S_2, \alpha_2)$ there is a DXA $A_\cup = ([S_1, S_2], \alpha_\cup)$ recognizing $L_\omega(A_1) \cup L_\omega(A_2)$
3) For every two DXAs $A_1 = (S_1, \alpha_1)$ and $A_2 = (S_2, \alpha_1)$ there is a DXA $A_\cup = ([S_1, S_2], \alpha_\cap)$ recognizing $L_\omega(A_1) \cap L_\omega(A_2)$

Beyond Trinities

- Rabin automata: 1): No. 2): Yes. 3): No.

- Given two DRAs $A_1 = (S_1, \alpha_1)$ and $A_2 = (S_2, \alpha_2)$, the DRA $A_\cup = ([S_1, S_2], \alpha)$ where

$$\alpha = \begin{array}{c} \{ \langle F_1 \times Q_2, G_1 \times Q_2 \rangle : \langle F_1, G_1 \rangle \in \alpha_1 \} \\ \cup \\ \{ \langle Q_1 \times F_2, Q_1 \times G_2 \rangle : \langle F_2, G_2 \rangle \in \alpha_2 \} \end{array}$$

recognizes $L_\omega(A_1) \cup L_\omega(A_2)$

Beyond Trinities

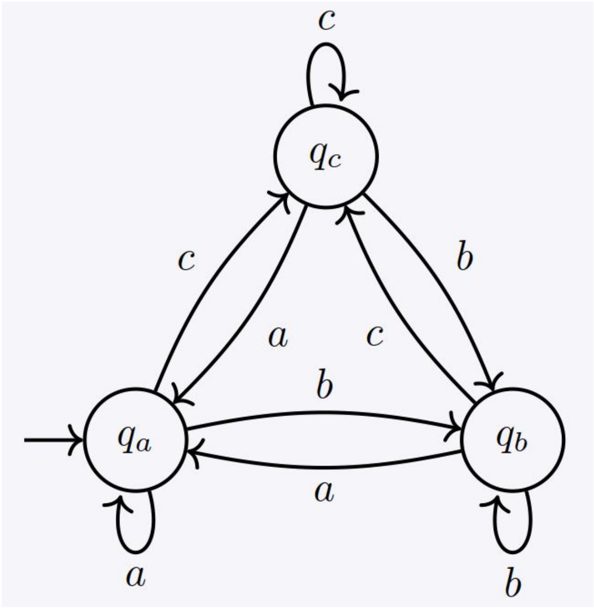
- Two further Trinities (see notes):
  - Street automata: 1): Yes. 2): No. 3): No.
  - Parity automata: 1): No. 2): No. 3): Yes.

- A final Trinity:
  - Muller automata: 1): Yes. 2): Yes. 3): Yes.

Muller automata

- Automata with arbitrary acceptance conditions.
- A Muller automaton (NMA) is an automaton $A = (S, \alpha)$ where $\alpha : 2^Q \to \{0,1\}$ is an arbitrary acceptance condition.
- We represent α by the set $\boldsymbol{\mathcal{F}}$ of all sets of states $Q' \subseteq Q$ such that $\alpha(Q') = 1$.
- A run is accepting if the set of states it visits infinitely often is equal to one of the sets in $\boldsymbol{\mathcal{F}}$.
- Theorem: RE, NMA, and DMA form a Trinity.

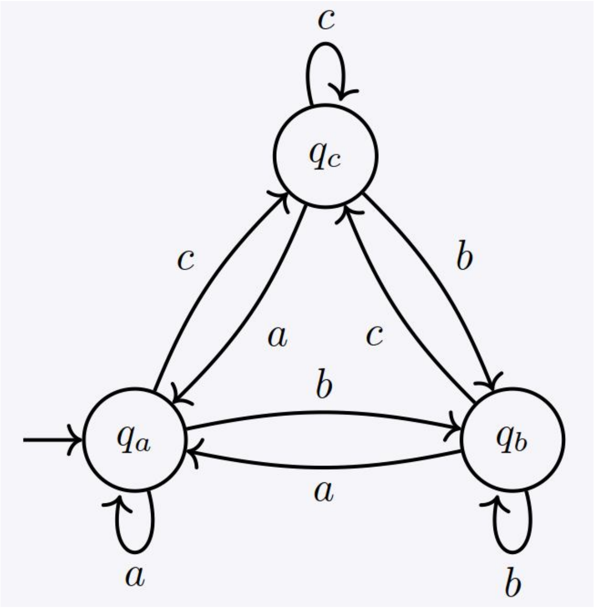
Muller automata



- Infinitely many a

$\{\ \{q_a\}, \{q_a, q_b\}, \{q_a, q_c\}, \{q_a, q_b, q_c\}\ \}$

Muller automata

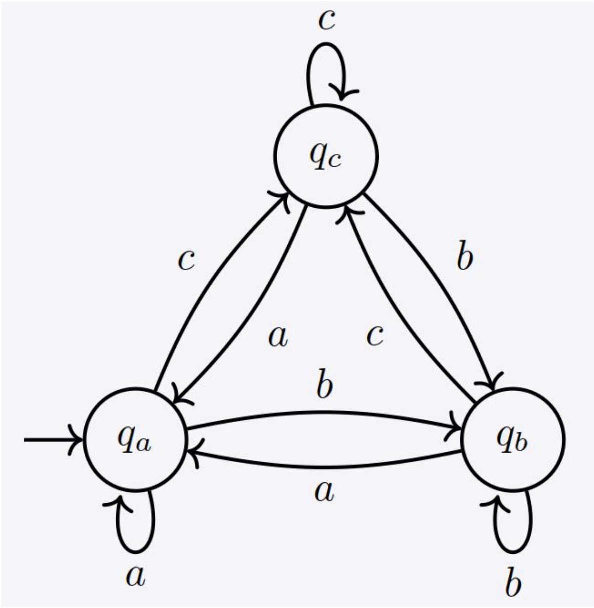


- Infinitely many a

  $\{\ \{q_a\}, \{q_a, q_b\}, \{q_a, q_c\}, \{q_a, q_b, q_c\}\ \}$

- Infinitely many a or infinitely many b

Muller automata



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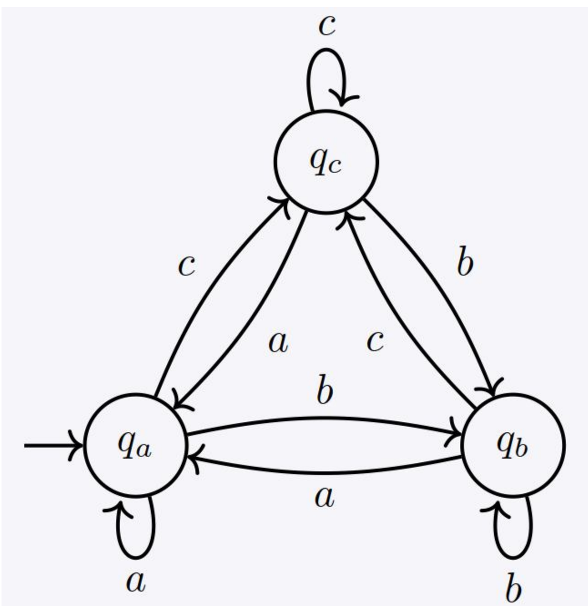
Muller automata



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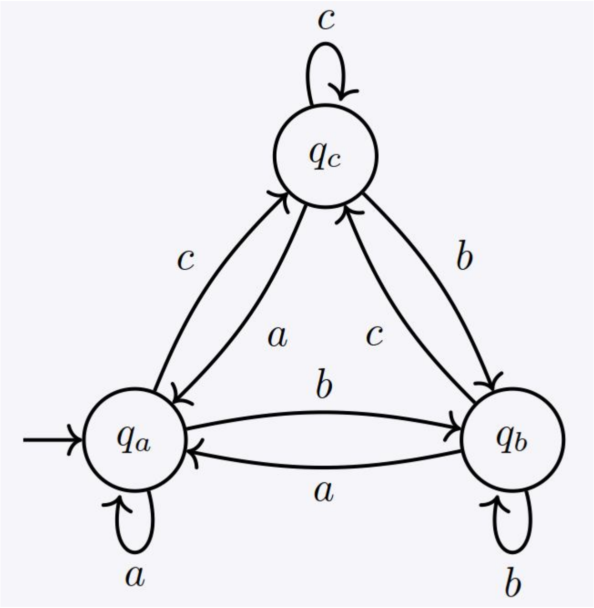
Muller automata



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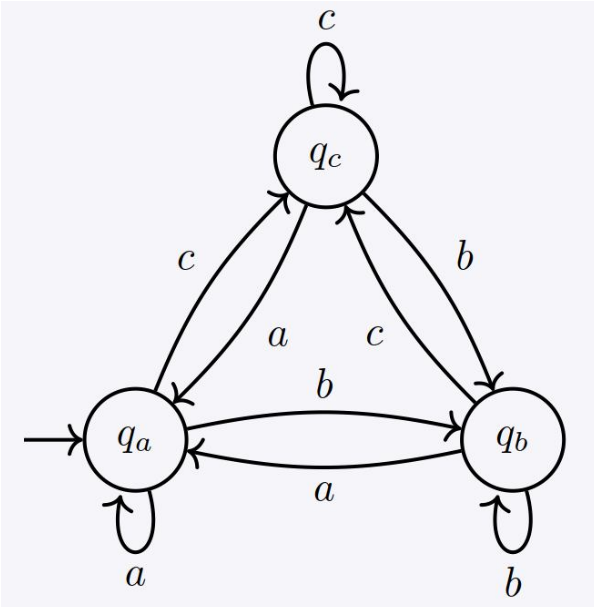
Muller automata



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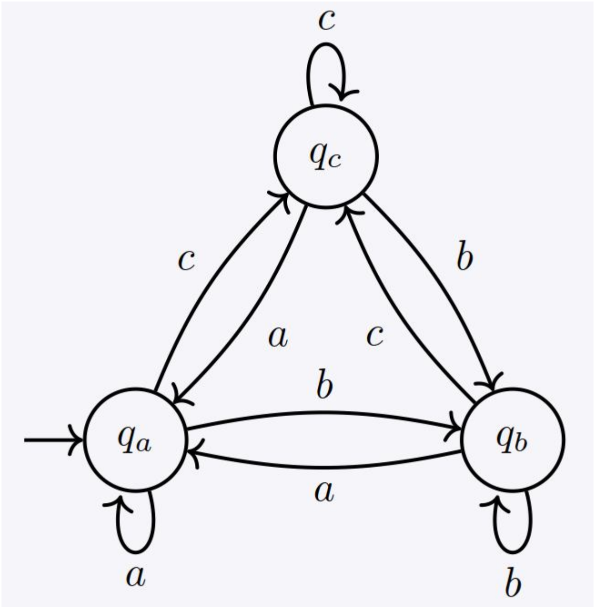
Muller automata



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Boolean operations on DMAs

- Let $A = (S, \mathcal{F})$ be a DMA. The DRA $\bar{A} = (S, \bar{\mathcal{F}})$, where

$$\bar{\mathcal{F}} = \{R \subseteq Q : R \notin \mathcal{F}\}$$

recognizes $\overline{L_\omega(A)}$.

Boolean operations on DMAs

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$$\bar{\mathcal{F}} = \{R \subseteq Q : R \notin \mathcal{F}\}$$

recognizes $\overline{L_\omega(A)}$.

Problem: $\bar{\mathcal{F}}$ can be exponentially larger than $\mathcal{F}$!!

Boolean operations on DMAs

- Let $A_1 = (S_1, \mathcal{F}_1)$ and $A_2 = (S_2, \mathcal{F}_2)$ be DMAs
- Given $R \subseteq Q_1 \times Q_2$, let $R|_1$ and $R|_2$ denote the projections of R on Q_1 and Q_2.
- The DRAs $A_\cup = ([S_1, S_2], \mathcal{F}_\cup)$ and $A_\cap = ([S_1, S_2], \mathcal{F}_\cap)$, where

  $$\mathcal{F}_\cup = \{R \subseteq Q_1 \times Q_2 : R|_1 \in \mathcal{F}_1 \text{ or } R|_2 \in \mathcal{F}_2\}$$

  $$\mathcal{F}_\cap = \{R \subseteq Q_1 \times Q_2 : R|_1 \in \mathcal{F}_1 \text{ and } R|_2 \in \mathcal{F}_2\}$$

  recognize $L_\omega(A_1) \cup L_\omega(A_2)$ and $L_\omega(A_1) \cap L_\omega(A_2)$.
- Same problem as for complementation: $\mathcal{F}_\cup$ and $\mathcal{F}_\cap$ can be exponentially larger than $\mathcal{F}$.

Summary

Automaton Type		Expr.	Det.	Union	Inters.	Comp.
NFA/DFA		<u>Y</u>	Y	<u>Y</u>	<u>Y</u>	<u>Y</u>
NBA/DBA	(Büchi)	<u>Y</u>	N	<u>Y</u>	N	N
NCA/DCA	(Co-Büchi)	N	Y	N	<u>Y</u>	N
NRA/DRA	(Rabin)	<u>Y</u>	Y	<u>Y</u>	N	N
NSA/DSA	(Streett)	<u>Y</u>	Y	N	<u>Y</u>	N
NPA/DPA	(Parity)	<u>Y</u>	Y	N	N	<u>Y</u>
NMA/DMA	(Muller)	Y	Y	Y	Y	Y

Expr: Is there a conversion from RE to NXA?
Det: Is there a conversion from NXA to DXA?
Union: Does pairing work for DXA and union?
Inters: Does pairing work for DXA and intersection?
Comp: Can DXA be complemented without changing the semi-automaton?

<u>Y</u>: the underlying conversion or operation has polynomial blow-up