Logical Methods in Computer Science
Volume 16, Issue 2, 2020, pp. 13:1–13:33
https://lmcs.episciences.org/

Submitted Nov. 15, 2018
Published Jun. 23, 2020

FORWARD ANALYSIS FOR WSTS, PART III: KARP-MILLER TREES *

MICHAEL BLONDIN [a], ALAIN FINKEL [b,c], AND JEAN GOUBAULT-LARRECQ [b]

[a] Université de Sherbrooke, Canada
e-mail address: michael.blondin@usherbrooke.ca

[b] Université Paris-Saclay, ENS Paris-Saclay, CNRS, Laboratoire Spécification et Vérification, France
e-mail address: {finkel,goubault}@lsv.fr

[c] Institut Universitaire de France

Abstract. This paper is a sequel of "Forward Analysis for WSTS, Part I: Completions" [STACS 2009, LZI Intl. Proc. in Informatics 3, 433–444] and "Forward Analysis for WSTS, Part II: Complete WSTS" [Logical Methods in Computer Science 8(3), 2012]. In these two papers, we provided a framework to conduct forward reachability analyses of WSTS, using finite representations of downward-closed sets. We further develop this framework to obtain a generic Karp-Miller algorithm for the new class of very-WSTS. This allows us to show that coverability sets of very-WSTS can be computed as their finite ideal decompositions. Under natural effectiveness assumptions, we also show that LTL model checking for very-WSTS is decidable. The termination of our procedure rests on a new notion of acceleration levels, which we study. We characterize those domains that allow for only finitely many accelerations, based on ordinal ranks.

1. Introduction

1.1. **Context.** A well-structured transition system (WSTS) is an infinite well-quasi-ordered set of states equipped with transition relations satisfying one of various possible monotonicity properties. WSTS were introduced in [Fin87] for the purpose of capturing properties common to a wide range of formal models used in verification. Since their inception, much of the work on WSTS has been dedicated to identifying generic classes of WSTS for which verification problems are decidable. Such problems include termination, boundedness [Fin87, Fin90, FPS01] and coverability [ACJT96, ACJT00, BFM17, BFM18]. In general, verifying safety and liveness properties corresponds respectively to deciding the coverability and the repeated

Key words and phrases: well-structured transition systems, Karp-Miller trees, model checking, coverability, ideals.

* Extended and expanded version of "Analysis for WSTS, Part III: Karp-Miller Trees" by M. Blondin, A. Finkel and J. Goubault-Larrecq, in Proc. 37$^{\text{th}}$ IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS), 2017.

M. Blondin was supported by the Fonds de recherche du Quebec – Nature et technologies (FRQNT) and by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (NSERC).

DOI:10.23638/LMCS-16(2:13)2020

control-state reachability problems. Coverability can be decided for WSTS by two different algorithms: the backward algorithm [ACJT96, ACJT00] and by combining two forward semi-procedures, one of which enumerates all downward-closed invariants [GRB06, BFM17, BFM18]. Repeated control-state reachability is undecidable for general WSTS, but decidable for Petri nets by use of the Karp-Miller coverability tree [KM67] and the detection of increasing sequences. That technique fails on well-structured extensions of Petri nets: generating the Karp-Miller tree does not always terminate on ν-Petri nets [RMdF11], on reset Petri nets [DFS98], on transfer Petri nets, on broadcast protocols, and on the depth-bounded π-calculus [HMM14, RM12, ZWH12] which can simulate reset Petri nets. This is perhaps why little research has been conducted on coverability tree algorithms and model checking of liveness properties for general WSTS. Nonetheless, some recent Petri nets extensions, e.g. ω-Petri nets [GHPR15] and unordered data Petri nets [HLL$^+$16], benefit from algorithms in the style of Karp and Miller. Hence, there is hope of finding a general framework of WSTS with Karp-Miller-like algorithms.

1.2. **The Karp-Miller coverability procedure.** In 1967, Karp and Miller [KM67] proposed what is now known as the Karp-Miller coverability tree algorithm, which computes a finite representation (the *clover*) of the downward closure (the *cover*) of the reachability set of a Petri net. In 1978, Valk extended the Karp-Miller algorithm to post-self-modifiying nets [Val78], a strict extension of Petri nets. In 1987, the second author proposed a generalization of the Karp-Miller algorithm that applies to a class of finitely branching WSTS with strong-strict monotonicity, and having a WSTS completion in which least upper bounds replace the original Petri nets ω-accelerations [Fin87, Fin90]. In 2004, Finkel, McKenzie and Picaronny [FMP04] applied the framework of [Fin90] to the construction of Karp-Miller trees for strongly increasing ω-recursive nets, a class generalizing post-self-modifiying nets. In 2005, Verma and the third author [VG05] showed that the construction of Karp-Miller trees can be extended to branching vector addition systems with states. In 2009, the second and the third authors [FG12] proposed a non-terminating procedure that computes the clover of *any* complete WSTS; this procedure terminates exactly on so-called cover-flattable systems. Recently, this framework has been used for defining computable accelerations in non-terminating Karp-Miller algorithms for both the depth-bounded π-calculus [HMM14] and for ν-Petri nets; terminating Karp-Miller trees are obtained for strict subclasses.

1.3. **Model checking WSTS.** In 1994, Esparza [Esp94] showed that model checking the linear time μ-calculus is decidable for Petri nets by using both the Karp-Miller algorithm and a decidability result due to Valk and Jantzen [VJ85] on infinite T-continual sequences in Petri nets. LTL is undecidable for Petri net extensions such as lossy channel systems [AJ94] and lossy counter machines [Sch10]. In 1998, Emerson and Namjoshi [EN98] studied the model checking of liveness properties for complete WSTS, but their procedure is not guaranteed to terminate. In 2004, Kouzmin, Shilov and Sokolov [KSS04] gave a generic computability result for a fragment of the μ-calculus; in 2006 and 2013, Bertrand and Schnoebelen [BBS06, BS13] studied fixed points in well-structured regular model checking; both [KSS04] and [BS13] are concerned with formulas with upward-closed atomic propositions, and do not subsume LTL. In 2011, Chambart, Finkel and Schmitz [CFS11, CFS16] showed that LTL is decidable for the recursive class of trace-bounded complete WSTS; a class which does not contain all Petri nets.

1.4. **Our contributions.**

- We define *very-well-structured transition systems (very-WSTS)*; a class defined in terms of WSTS completions, and which encompasses models such as Petri nets, ω-Petri nets, post-self-modifying nets and strongly increasing ω-recursive nets. We show that coverability sets of very-WSTS are computable as finite sets of ideals.
- The general clover algorithm of [FG12], based on the ideal completion studied in [FG09], does not necessarily terminate and uses an abstract acceleration enumeration. We give an algorithm, the Ideal Karp-Miller algorithm, which organizes accelerations within a tree. We show that this algorithm terminates under natural order-theoretic and effectiveness conditions, which we make explicit. This allows us to unify various versions of Karp-Miller algorithms in particular classes of WSTS.
- We identify the crucial notion of *acceleration level* of an ideal, and relate it to ordinal ranks of sets of reachable states in the completion. We show, notably, that termination is equivalent to the rank being strictly smaller than ω^2. This classifies WSTS into those with high rank (the bad ones), among which those whose sets of states consist of words (e.g., lossy channel systems) or multisets; and those with low rank (the good ones), among which Petri nets and post-self-modifying nets.
- We show that the downward closure of the trace language of a very-WSTS is computable, again as a finite union of ideals. This shows that downward trace inclusion is decidable for very-WSTS.
- Finally, we prove the decidability of model checking liveness properties for very-WSTS under some effectiveness hypotheses.

1.5. **A short story of well-structured transition systems.** *Structured transition systems* were initially defined and studied in [Fin86, Fin87, Fin90] as monotone transition systems equipped with a well-quasi-ordering on their set of states. Termination was shown decidable for *structured transition systems* with *transitive* monotonicity, while boundedness was shown decidable for structured transition systems with *strict* monotonicity. For a subclass of finitely branching labeled structured transition systems with strong-strict monotonicity, initially called *well structured transition systems* in [Fin86, Fin87, Fin90], a generalization of the Karp-Miller algorithm was shown to compute their coverability sets. In [ACJT96, ACJT00], the coverability problem was shown to be decidable for *well-structured systems* [ACJT96, Def. 3.1], i.e. *labeled* structured transition systems with *strong monotonicity* and satisfying an additional *effective hypothesis*: the existence of an algorithm to compute the finite set of minimal elements $\min(\mathrm{Pre}\,(\uparrow s))$, where $\mathrm{Pre}\,(\uparrow s)$ is the set of immediate predecessors of the upward-closure $\uparrow s$ of a state s. In [FPS01], mathematical properties were distinguished from effective properties, and the coverability problem was shown decidable for the *entire* class of structured transition systems satisfying the additional *effective hypothesis* that there exists an algorithm to compute the finite set $\min(\uparrow \mathrm{Pre}\,(\uparrow s))$, i.e. the hypotheses of transitions labeling and strong monotonicity made in [ACJT96] turned out to be superfluous.

Today, following the presentation of [FPS01], what is *mathematically* known as *well structured transition systems* is exactly the original class of *structured transition systems*; and necessary effective hypotheses are added for obtaining decidability of properties such as termination, coverability and boundedness.

1.6. **Differences between very-WSTS and WSTS of** [Fin90]**.** The class of WSTS of [Fin90, Def. 4.17] is reminiscent of very-WSTS. It requires WSTS to be finitely branching and strictly monotone, whereas our definition allows infinite branching and requires the *completion* to be strictly monotone. Moreover, [Fin90, Thm. 4.18], which claims that its Karp-Miller procedure terminates, is incorrect since it does not terminate on transfer Petri nets and broadcast protocols [EFM99], which are finitely branching and strictly monotone WSTS. Finally, some assumptions required to make the Karp-Miller procedure of [Fin90] effective are missing.

2. Preliminaries

We write $\subseteq$ for set inclusion and $\subset$ for strict set inclusion. A relation $\leq \subseteq X \times X$ over a set X is a *quasi-ordering* if it is reflexive and transitive, and a *partial ordering* if it is antisymmetric as well. It is *well-founded* if it has no infinite descending chain. A quasi-ordering $\leq$ is a *well-quasi-ordering* (resp. *well partial order*), *wqo* (resp. *wpo*) for short, if for every infinite sequence $x_0, x_1, \cdots \in X$, there exist $i < j$ such that $x_i \leq x_j$. This is strictly stronger than being well-founded.

One example of well-quasi-ordering is the componentwise ordering of tuples over $\mathbb{N}$. More formally, $\mathbb{N}^d$ is well-quasi-ordered by $\leq$ where, for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^d$, $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if $\boldsymbol{x}(i) \leq \boldsymbol{y}(i)$ for every $i \in [d]$. We extend $\mathbb{N}$ to $\mathbb{N}_\omega \stackrel{\text{def}}{=} \mathbb{N} \cup \{\omega\}$ where $n \leq \omega$ for every $n \in \mathbb{N}_\omega$. $\mathbb{N}_\omega^d$ ordered componentwise is also well-quasi-ordered. Let Σ be a finite alphabet. We write Σ^*, Σ^+ and Σ^ω to denote the set of finite words, nonempty finite words and infinite words over Σ, respectively. For every (finite or infinite) nonempty word w, we write w_i to denote its i^{th} letter. For every $u, v \in \Sigma^*$, we write $u \preceq v$ if u is a subword of v, i.e. u can be obtained from v by removing zero, one or multiple letters. Σ^* is well-quasi-ordered by $\preceq$.

2.1. **Transition systems.** A *(labeled) transition system* is a triple $\mathcal{S} = (X, \xrightarrow{\Sigma})$ such that X is a set, Σ is a finite alphabet, and $\xrightarrow{a} \subseteq X \times X$ for every $a \in \Sigma$. Elements of X are called the *states* of $\mathcal{S}$, and each $\xrightarrow{a}$ is a *transition relation* of $\mathcal{S}$. A *class $\mathcal{C}$ of transition systems* is any set of transition systems. We extend transition relations to sequences over Σ, i.e. for every $x, y \in X$, $x \xrightarrow{\varepsilon} x$, and $x \xrightarrow{wa} y$ if there exists $x' \in X$ such that $x \xrightarrow{w} x' \xrightarrow{a} y$. We write $x \xrightarrow{*} y$ (resp. $x \xrightarrow{+} y$) if there exists $w \in \Sigma^*$ (resp. $w \in \Sigma^+$) such that $x \xrightarrow{w} y$. The finite and infinite *traces* of a transition system $\mathcal{S}$ from a state $x \in X$ are respectively defined as

$$\text{Traces}_{\mathcal{S}}(x) \stackrel{\text{def}}{=} \{w \in \Sigma^* : x \xrightarrow{w} y \text{ for some } y \in X\}, \text{ and}$$

$$\omega\text{-Traces}_{\mathcal{S}}(x) \stackrel{\text{def}}{=} \{w \in \Sigma^\omega : x \xrightarrow{w_1} x_1 \xrightarrow{w_2} \cdots \text{ for some } x_1, x_2, \ldots \in X\}.$$

We define the *immediate successors* and *immediate predecessors* of a state x under some sequence $w \in \Sigma^*$ as

$$\text{Post}_{\mathcal{S}}(x, w) \stackrel{\text{def}}{=} \{y \in X : x \xrightarrow{w} y\}, \text{ and}$$

$$\text{Pre}_{\mathcal{S}}(x, w) \stackrel{\text{def}}{=} \{y \in X : y \xrightarrow{w} x\}.$$

The *successors* and *predecessors* of $x \in X$ are

$$\text{Post}^*_{\mathcal{S}}(x) \stackrel{\text{def}}{=} \{y \in X : x \xrightarrow{*} y\}, \text{ and}$$

$$\text{Pre}^*_{\mathcal{S}}(x) \stackrel{\text{def}}{=} \{y \in X : y \xrightarrow{*} x\}.$$

These notations are naturally extended to sets, e.g. $\mathrm{Post}_{\mathcal{S}}(A, w) \stackrel{\text{def}}{=} \bigcup_{x \in A} \mathrm{Post}_{\mathcal{S}}(x, w)$.

We say that $\mathcal{S}$ is *deterministic* if $|\mathrm{Post}_{\mathcal{S}}(x, a)| \leq 1$ for every $x \in X$ and $a \in \Sigma$. When $\mathcal{S}$ is deterministic, each $a \in \Sigma$ induces a partial function $t_a : X \to X$ such that $t_a(x) = y$ for each $x \in X$ such that $\mathrm{Post}_{\mathcal{S}}(x, a) = \{y\}$. For readability, we simply write a for t_a, i.e. $a(x) = t_a(x)$. For every $w \in \Sigma^*$, we write $w(x)$ for $\mathrm{Post}_{\mathcal{S}}(x, w)$ if $\mathrm{Post}_{\mathcal{S}}(x, w) \neq \emptyset$.

2.2. **Well-structured transition systems.** An *ordered (labeled) transition system* is a triple $(X, \xrightarrow{\Sigma}, \leq)$ such that $(X, \xrightarrow{\Sigma})$ is a *(labeled) transition system* and $\leq$ is a quasi-ordering. An ordered transition system $\mathcal{S}$ is a *well-structured transition system (WSTS)* if $\leq$ is a well-quasi-ordering and $\mathcal{S}$ is *monotone*, i.e. for all $x, x', y \in X$ and $a \in \Sigma$ such that $x \xrightarrow{a} y$ and $x' \geq x$, there exists $y' \in X$ such that $x' \xrightarrow{*} y'$ and $y' \geq y$. Many other types of monotonicities were defined in the literature (see e.g. [FPS01]), but, for our purposes, we only need to introduce strong monotonicities. We say that $\mathcal{S}$ has *strong monotonicity* if for all $x, x', y \in X$ and $a \in \Sigma$, $x \xrightarrow{a} y$ and $x' \geq x$ implies $x' \xrightarrow{a} y'$ for some $y' \geq y$. We say that $\mathcal{S}$ has *strong-strict monotonicity*[1] if it has strong monotonicity and for all $x, x', y \in X$ and $a \in \Sigma$, $x \xrightarrow{a} y$ and $x' > x$ implies $x' \xrightarrow{a} y'$ for some $y' > y$.

Remark. Although the coverability problem is decidable for *unlabeled* WSTS, we consider labeled WSTS here for two main reasons: firstly, we study the traces of WSTS and their model checking, hence their transitions must be labeled with a finite alphabet; secondly, we extend the acceleration technique to compute the downward closure of reachability sets: we need a labeling of transitions to properly define the acceleration of a sequence of transitions (this labeling is not necessary for Petri nets, but in an abstract model like WSTS, the labeling seems necessary).

2.3. **Verification problems.** We say that a *target state* $y \in X$ is *coverable from* an *initial state* $x \in X$ if there exists $z \geq y$ such that $x \xrightarrow{*} z$. The *coverability problem* asks whether a target state y is coverable from an initial state x. The *repeated coverability problem* asks whether a target state y is coverable infinitely often from an initial state x; i.e. whether there exist $z_0, z_1, \cdots \in X$ such that $x \xrightarrow{*} z_0 \xrightarrow{+} z_1 \xrightarrow{+} \cdots$ and $z_i \geq y$ for every $i \in \mathbb{N}$.

3. An investigation of the Karp-Miller algorithm

In order to present our Karp-Miller algorithm for WSTS, we first highlight the key components of the Karp-Miller algorithm for Petri nets. A *Petri net* with d places is a WSTS $\mathcal{V} = (\mathbb{N}^d, \xrightarrow{T}, \leq)$ induced by a finite set $T \subseteq \mathbb{N}^d \times \mathbb{N}^d$ and the rules:

$$\boldsymbol{x} \xrightarrow{t} \boldsymbol{y} \iff \boldsymbol{x} \geq \boldsymbol{pre} \wedge \boldsymbol{y} = \boldsymbol{x} - \boldsymbol{pre} + \boldsymbol{post} \quad \text{for every } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^d, t = (\boldsymbol{pre}, \boldsymbol{post}) \in T.$$

Petri nets are deterministic and have strong-strict monotonicity. Given a Petri net with d places and a vector $\boldsymbol{x}_{\text{init}} \in \mathbb{N}^d$, the Karp-Miller algorithm initializes a rooted tree whose root is labeled by $\boldsymbol{x}_{\text{init}}$. For every $(\boldsymbol{pre}, \boldsymbol{post}) \in T$ such that $\boldsymbol{x} \geq \boldsymbol{pre}$, a child labeled by $\boldsymbol{x} - \boldsymbol{pre} + \boldsymbol{post}$ is added to the root. This process is repeated successively to the new nodes. If a newly added node $c : \boldsymbol{x}$ has an ancestor $c' : \boldsymbol{x}'$ such $\boldsymbol{x} = \boldsymbol{x}'$, then it is not

[1] Strong-strict monotonicity should not be confused with strong *and* strict monotonicities. Here strongness and strictness have to hold at the *same* time.

explored furthermore. If a newly added node $c : \boldsymbol{x}$ has an ancestor $c' : \boldsymbol{x}'$ such $\boldsymbol{x} > \boldsymbol{x}'$, then c is relabeled by the vector $\boldsymbol{y} \in \mathbb{N}_\omega^d$ such that $\boldsymbol{y}(i) \stackrel{\text{def}}{=} \boldsymbol{x}(i)$ if $\boldsymbol{x}(i) = \boldsymbol{x}'(i)$ and $\boldsymbol{y}(i) \stackrel{\text{def}}{=} \omega$ if $\boldsymbol{x}(i) > \boldsymbol{x}'(i)$. The latter operation is called an *acceleration* of c with respect to c'.

A vector $\boldsymbol{x}_{\text{tgt}}$ is coverable from $\boldsymbol{x}_{\text{init}}$ if and only if the resulting tree $\mathcal{T}$ contains a node $c : \boldsymbol{x}$ such that $\boldsymbol{x} \geq \boldsymbol{x}_{\text{tgt}}$. A slightly more complex characterization in terms of $\mathcal{T}$ further allows to decide whether a vector $\boldsymbol{x}_{\text{tgt}}$ is repeatedly coverable from $\boldsymbol{x}_{\text{init}}$.

3.1. **Ideals and completions.** One feature of the Karp-Miller algorithm is that it works over $\mathbb{N}_\omega^d$ instead of $\mathbb{N}^d$. Intuitively, vectors containing some ω correspond to "limit" elements. For a generic WSTS $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$, a similar extension of X is not obvious. Let us present one, called the *completion* of $\mathcal{S}$ in [FG12]. Instead of operating over X, the completion of $\mathcal{S}$ operates over the so-called *ideals* of X. In particular, the ideals of $\mathbb{N}^d$ are isomorphic to $\mathbb{N}_\omega^d$.

Let X be a set quasi-ordered by $\leq$. The *downward closure* of $D \subseteq X$ is defined as

$$\downarrow D \stackrel{\text{def}}{=} \{x \in X : x \leq y \text{ for some } y \in D\}.$$

A subset $D \subseteq X$ is *downward-closed* if $D = \downarrow D$. An *ideal* is a downward-closed subset $I \subseteq X$ that is additionally *directed*: I is non-empty and for all $x, y \in I$, there exists $z \in I$ such that $x \leq z$ and $y \leq z$ (equivalently, every finite subset of I has an upper bound in I). We denote the set of ideals of X by $\text{Idl}(X)$, i.e. $\text{Idl}(X) \stackrel{\text{def}}{=} \{D \subseteq X : D = \downarrow D \text{ and } D \text{ is directed}\}$.

It is known that

$$\text{Idl}(\mathbb{N}^d) = \{A_1 \times \cdots \times A_d : A_1, \ldots, A_d \in \{\downarrow n : n \in \mathbb{N}\} \cup \{\mathbb{N}\}\}.$$

Therefore, every ideal of $\mathbb{N}^d$ is naturally represented by some vector of $\mathbb{N}_\omega^d$, and vice versa. We write $\omega\text{-rep}(I)$ for this representation, for every $I \in \text{Idl}(\mathbb{N}^d)$. For example, the ideal $I = \mathbb{N} \times \downarrow 8 \times \downarrow 3 \times \mathbb{N}$ is represented by $\omega\text{-rep}(I) = (\omega, 8, 3, \omega)$.

Downward-closed subsets can often be represented by finitely many ideals: in fact, the following Theorem 3.1 gives a complete characterization of quasi-ordered sets for which every downward closed subset is equal to a finite union of ideals.

Theorem 3.1 ([ET43, Bon75, Pou79, PZ85, Fra86, LMP87, BFM17])**.** *A countable quasi-ordered set X contains no infinite antichain if, and only if, every downward closed subset of X is equal to a finite union of ideals.*

From this theorem, we immediately deduce a (known) corollary for wqos:

Corollary 3.2. *Let X be a well-quasi-ordered set. For every downward-closed subset $D \subseteq X$, there exist $I_1, I_2, \ldots, I_n \in \text{Idl}(X)$ such that $D = I_1 \cup I_2 \cup \cdots \cup I_n$.*

The existence of such a decomposition has been proved numerous times (for partial orderings instead of quasi-orderings) in the order theory community [Bon75, Pou79, PZ85, Fra86, LMP87] under different terminologies, and is a particular case of a more general set theory result of Erdős and Tarski [ET43] on the existence of *limit numbers* between $\aleph_0$ and $2^{\aleph_0}$. The paper [FM14] explains in detail the fact that Theorem 3.1 is attributed to Erdős and Tarski because the difficult direction (left to right) of Theorem 3.1 can be deduced from [ET43, Theorem 1]. For the reader interested in a simple and self-contained proof, we refer to [BFM17, Theorem 3.3]. More specifically, this proof is based on the fact that such decompositions exist in well-quasi-ordered sets and is reminiscent of Fraïssé's proof strategy [Fra86, Sect. 4.7.2, p. 124], which is based on [Bon75, Lemma 2, p. 193].

Theorem 3.1 gives rise to a canonical decomposition of downward-closed sets. The *ideal decomposition* of a downward-closed subset $D \subseteq X$ is the set of maximal ideals contained in D w.r.t. inclusion. We denote the ideal decomposition of D by $\mathrm{IdealDecomp}(D) \stackrel{\text{def}}{=} \max_{\subseteq}\{I \in \mathrm{Idl}(X) : I \subseteq D\}$. By Corollary 3.2, $\mathrm{IdealDecomp}(D)$ is finite, and $D = \bigcup_{I \in \mathrm{IdealDecomp}(D)} I$. In [FG12, BFM18], the notion of ideal decomposition is used to define the completion of unlabeled WSTS. We slightly extend this notion to labeled WSTS:

Definition 3.3. Let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a labeled WSTS. The *completion* of $\mathcal{S}$ is the labeled transition system $\widehat{\mathcal{S}} = (\mathrm{Idl}(X), \overset{\Sigma}{\rightsquigarrow}, \subseteq)$ such that

$$I \overset{a}{\rightsquigarrow} J \iff J \in \mathrm{IdealDecomp}(\downarrow \mathrm{Post}_{\mathcal{S}}(I, a)).$$

The completion of a WSTS enjoys numerous properties. In particular, it has strong monotonicity, and it is finitely branching [BFM18], i.e. $\mathrm{Post}_{\widehat{\mathcal{S}}}(I, a)$ is finite for every $I \in \mathrm{Idl}(X)$ and $a \in \Sigma$. Note that if $\mathcal{S}$ has strong-strict monotonicity, then this property is not necessarily preserved by $\widehat{\mathcal{S}}$ [BFM18]. Moreover, the completion of a WSTS may not be a WSTS since $\mathrm{Idl}(X)$ is not always well-quasi-ordered by $\subseteq$. However, for the vast majority of models used in formal verification, $\mathrm{Idl}(X)$ is well-quasi-ordered, and hence completions remain well-structured. Indeed, $\mathrm{Idl}(X)$ is well-quasi-ordered if and only if X is a so-called ω^2-wqo, and widespread wqos, except possibly graphs under minor embedding, are ω^2-wqo, as discussed in [FG12]. The traces of a WSTS are closely related to those of its completion:

Proposition 3.4 ([BFM18])**.** *The following holds for every WSTS $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$:*

(1) *For all $x, y \in X$ and $w \in \Sigma^*$, if $x \xrightarrow{w} y$, then for every ideal $I \supseteq \downarrow x$, there exists an ideal $J \supseteq \downarrow y$ such that $I \overset{w}{\rightsquigarrow} J$.*
(2) *For all $I, J \in \mathrm{Idl}(X)$ and $w \in \Sigma^*$, if $I \overset{w}{\rightsquigarrow} J$, then for every $y \in J$, there exist $x \in I, y' \in X$ and $w' \in \Sigma^*$ such that $x \xrightarrow{w'} y'$ and $y' \geq y$. If $\mathcal{S}$ has strong monotonicity, then $w' = w$.*
(3) *if $\mathcal{S}$ has strong monotonicity, then $\bigcup_{J \in Post_{\widehat{\mathcal{S}}}(I,w)} J = \downarrow Post_{\mathcal{S}}(I, w)$ for all $I \in \mathrm{Idl}(X)$ and $w \in \Sigma^*$.*
(4) *if $\mathcal{S}$ has strong monotonicity, then $\mathrm{Traces}_{\mathcal{S}}(x) = \mathrm{Traces}_{\widehat{\mathcal{S}}}(\downarrow x)$ and $\omega\text{-}\mathrm{Traces}_{\mathcal{S}}(x) \subseteq \omega\text{-}\mathrm{Traces}_{\widehat{\mathcal{S}}}(\downarrow x)$ for every $x \in X$.*

Proof.

(1–3) The proofs given in [BFM18] for unlabeled WSTS can be adapted straightforwardly to labeled WSTS. For completeness, these adaptations are given in the appendix.

(4)
- For every $w \in \mathrm{Traces}_{\mathcal{S}}(x)$, there is a state y such that $x \xrightarrow{w} y$. Use (1) on $I = \downarrow x$: we obtain an ideal J such that $I \overset{w}{\rightsquigarrow} J$, showing that $w \in \mathrm{Traces}_{\widehat{\mathcal{S}}}(\downarrow x)$. Conversely, for every $w \in \mathrm{Traces}_{\widehat{\mathcal{S}}}(\downarrow x)$, there is an ideal J such that $I \overset{w}{\rightsquigarrow} J$, where $I = \downarrow x$. Ideals are non-empty, so pick $y \in J$. By (2), there are states $x' \in I$ and $y' \geq y$ such that $x' \xrightarrow{w} y'$. The fact that x' is in I, namely that $x' \leq x$, allows us to invoke strong monotonicity and obtain a state $y'' \geq y'$ such that $x \xrightarrow{w} y''$. In particular, w is in $\mathrm{Traces}_{\mathcal{S}}(x)$.
- Let $w \in \omega\text{-}\mathrm{Traces}_{\Sigma}(x)$. Let $x_0 \stackrel{\text{def}}{=} x$, and let $x_1, x_2, \ldots \in X$ be such that $x \xrightarrow{w_1} x_1 \xrightarrow{w_2} x_2 \xrightarrow{w_3} \cdots$. Let $I_0 \stackrel{\text{def}}{=} \downarrow x$. By (1), there exists an ideal $I_1 \supseteq \downarrow x_1$ such that

$I_0 \overset{w_1}{\rightsquigarrow} I_1$. This process can be repeated using (1) to obtain $I_{i-1} \overset{w_i}{\rightsquigarrow} I_i$ with $I_i \supseteq {\downarrow} x_i$ for every $i > 0$. □

It is worth noting that if $\mathcal{S}$ is infinitely branching, then an infinite trace of $\widehat{\mathcal{S}}$ from ${\downarrow} x$ is not necessarily an infinite trace of $\mathcal{S}$ from x (e.g. see [BFM18]).

Whenever the completion of a WSTS $\mathcal{S}$ is deterministic, we will often write $w(I)$ for $\mathrm{Post}_{\widehat{\mathcal{S}}}(I, w)$ if the latter is nonempty and if there is no ambiguity with $\mathrm{Post}_{\mathcal{S}}(I, w)$.

3.2. **Levels of ideals.** The Karp-Miller algorithm terminates for the following reasons: $\mathbb{N}_\omega^d$ is well-quasi-ordered and ω's can only be added to vectors along a branch at most d times. Loosely speaking, the latter property means that $\mathrm{Idl}(\mathbb{N}^d)$ has $d+1$ "levels". Here, we generalize this notion. We say that an infinite sequence of ideals $I_0, I_1, \ldots \in \mathrm{Idl}(X)$ is an *acceleration candidate* if $I_0 \subset I_1 \subset \cdots$. An acceleration candidate *is below* $J \in \mathrm{Idl}(X)$ if $I_i \subseteq J$ for every $i \in \mathbb{N}$, and it *goes through* a set $A \subseteq \mathrm{Idl}(X)$ if $I_i \in A$ for some $i \in \mathbb{N}$.

Definition 3.5. The n^{th} *level* of $\mathrm{Idl}(X)$ is defined as

$$A_n(\mathrm{Idl}(X)) = \begin{cases} \emptyset, & n = 0, \\ \{I \in \mathrm{Idl}(X) : \text{every accel. candidate below } I \text{ goes through } A_{n-1}\} & n > 0. \end{cases}$$

When X is clear from the context, we will simply write A_n instead of $A_n(\mathrm{Idl}(X))$. For the specific case of $X = \mathbb{N}^d$, it can be shown that:

$$A_1 = \{I \in \mathrm{Idl}(\mathbb{N}^d) : \omega\text{-}rep(I) \textit{ has strictly less than } 1 \textit{ occurrence of } \omega\},$$
$$A_2 = \{I \in \mathrm{Idl}(\mathbb{N}^d) : \omega\text{-}rep(I) \textit{ has strictly less than } 2 \textit{ occurrences of } \omega\},$$
$$\vdots$$

Hence, for all $n \geq 0$:

$$A_n = \{I \in \mathrm{Idl}(\mathbb{N}^d) : \omega\text{-}rep(I) \textit{ has strictly less than } n \textit{ occurrences of } \omega\}.$$

Therefore, we have $\emptyset \subset A_1 \subset \cdots \subset A_{d+1} = A_{d+2} = \cdots$ which corresponds to the fact that $\mathrm{Idl}(\mathbb{N}^d)$ has $d+1$ different levels. In particular, if we identify A_{d+1} with $\mathbb{N}_\omega^d$, i.e. the set of its ω-representations, then A_{d+k} is equivalent to $\mathbb{N}_\omega^d$ for every $k \geq 1$. More formally:

Proposition 3.6. *$A_n(\mathbb{N}_\omega^d)$ is the set of d-tuples with less than n components equal to ω.*

Proof. Using the fact that $A_n(\mathbb{N}_\omega^d)$ grows as n grows, it suffices to show the claim for $n \leq d+1$. This is shown by induction on n. The case $n = 0$ is obvious.

Let $1 \leq n \leq d+1$. If $\boldsymbol{x} \in \mathbb{N}_\omega^d$ has at least n components equal to ω, we obtain an acceleration candidate by picking an index j such that $\boldsymbol{x}(j) = \omega$, and forming the tuples $(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(j-1), i, \boldsymbol{x}(j+1), \ldots, \boldsymbol{x}(d))$ for $i \in \mathbb{N}$. By induction hypothesis, these tuples have at least $n-1$ components equal to ω and therefore cannot be in $A_{n-1}(\mathbb{N}_\omega^d)$. This entails that $\boldsymbol{x}$ cannot be in $A_n(\mathbb{N}_\omega^d)$.

Conversely, assume that $\boldsymbol{x}$ has less than n components equal to ω, say at positions $1, 2, \ldots, k < n$ (the general case is obtained by applying a permutation of the indices). There are only finitely many tuples $\boldsymbol{y} \leq \boldsymbol{x}$ that have their first k components equal to ω. Therefore any acceleration candidate below $\boldsymbol{x}$, being infinite, must contain a tuple with at most $k-1$ components equal to ω. Since $k - 1 < n - 1$, by induction hypothesis it must go through $A_{n-1}(\mathbb{N}_\omega^d)$, showing that $\boldsymbol{x} \in A_n(\mathbb{N}_\omega^d)$. □

In general, we observe that ideal levels are monotonic and downward-closed with respect to ideal inclusion:

Proposition 3.7. *The following holds for every $n \in \mathbb{N}$:*
(1) *for every $I, J \in \mathrm{Idl}(X)$, if $I \in A_n$ and $J \subseteq I$, then $J \in A_n$,*
(2) *$A_n \subseteq A_{n+1}$.*

Proof. Let $n \in \mathbb{N}$. If $A_n = \emptyset$, then both claims follow immediately. Therefore, let us assume that $A_n \neq \emptyset$.

(1) Let $I \in A_n$ and let $J \in \mathrm{Idl}(X)$ be such that $J \subseteq I$. We must show that $J \in A_n$. Let $J_0, J_1, \dots$ be an acceleration candidate below J. We have $J_i \subseteq J \subseteq I$ for every $i \in \mathbb{N}$. Therefore, $J_0, J_1, \dots$ is also below I. Since $I \in A_n$, we conclude that $J_0, J_1, \dots$ goes through A_{n-1}, and hence that $J \in A_n$.

(2) Let $I \in A_n$. For the sake of contradiction, suppose $I \notin A_{n+1}$. By assumption, there exists an acceleration candidate $I_0, I_1, \dots$ below I that does not go through A_n. Note that $I_i \subseteq I$ for every $i \in \mathbb{N}$. By (1), this implies that $I_i \in A_n$ for every $i \in \mathbb{N}$. Therefore, we conclude that $I_0, I_1, \dots$ goes through A_n, which is a contradiction. □

We have seen that $\mathrm{Idl}(\mathbb{N}^d)$ has only $d+1$ levels, i.e. $A_{d+1}(\mathrm{Idl}(\mathbb{N}^d)) = \mathrm{Idl}(\mathbb{N}^d)$. We generalize this notion as follows:

Definition 3.8. $\mathrm{Idl}(X)$ has *finitely many levels* if there exists $n \in \mathbb{N}$ such that $A_n = \mathrm{Idl}(X)$.

In the forthcoming sections, we will be interested in sets of ideals that have finitely many levels. It is however worth mentioning that there are natural sets X whose ideals do not have finitely many levels of ideals, even if $\mathrm{Idl}(X)$ is assumed to be countable and well-quasi-ordered. We postpone this discussion to Section 6 where we will study ideal levels in more details and in a more abstract setting.

3.3. **Accelerations.** The last key aspect of the Karp-Miller algorithm is the possibility to accelerate nodes. In order to generalize this notion, let us briefly develop some intuition. Recall that a newly added node $c\colon \boldsymbol{x}$ is accelerated if it has an ancestor $c'\colon \boldsymbol{x}'$ such that $\boldsymbol{x} > \boldsymbol{x}'$. Consider the non-empty sequence w labeling the path from c' to c. Since Petri nets have strong-strict monotonicity, both over $\mathbb{N}^d$ and $\mathbb{N}_\omega^d$, $w^n(\boldsymbol{x})$ is defined for every $n \in \mathbb{N}$. For example, if $(5,0,1) \xrightarrow{w} (5,1,3)$ is encountered, $(5,1,3)$ is replaced by $(5,\omega,\omega)$. This represents the fact that for every $n \in \mathbb{N}$, there exists some reachable marking $\boldsymbol{y} \geq (5,n,n)$. Note that an acceleration increases the number of occurrences of ω. In our example, the ideal $I = {\downarrow}5 \times {\downarrow}1 \times {\downarrow}3$, which is of level 0, is replaced by $I' = {\downarrow}5 \times \mathbb{N} \times \mathbb{N}$, which is of level 2. Based on these observations, we extend the notion of acceleration to completions:

Definition 3.9. Let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a WSTS such that $\widehat{\mathcal{S}}$ is deterministic, let $I \in \mathrm{Idl}(X)$, and let $w \in \Sigma^+$ be such that $\mathrm{Post}_{\widehat{\mathcal{S}}}(I, w) \neq \emptyset$. The *acceleration* of I under w is defined as:

$$w^\infty(I) \stackrel{\text{def}}{=} \begin{cases} \bigcup_{k \in \mathbb{N}} w^k(I) & \text{if } I, w(I), w^2(I), \dots \text{ is an acceleration candidate,} \\ I & \text{otherwise.} \end{cases}$$

In other words, if I can be accelerated by repeatedly applying w, then its acceleration is the least upper bound of $I \subset w(I) \subset w^2(I) \subset \cdots$. This least upper bound is also an ideal:

Proposition 3.10. *Let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a WSTS such that $\widehat{\mathcal{S}}$ is deterministic. We have $w^\infty(I) \in \mathrm{Idl}(X)$ for every $I \in \mathrm{Idl}(X)$ and $w \in \Sigma^+$ such that $Post_{\widehat{\mathcal{S}}}(I, w) \neq \emptyset$.*

Proof. If $w^\infty(I) = I$, then the claim trivially holds. Thus, we may assume that $I, w(I)$, $w^2(I), \ldots$ is an acceleration candidate. Since $w^\infty(I)$ is a union of downward-closed sets, it is readily seen to be downward-closed. Let us show that it is also directed. Let $x, y \in w^\infty(I)$. There exist $k, \ell \in \mathbb{N}$ such that $x \in w^k(I)$ and $y \in w^\ell(I)$. Therefore, both x and y are elements of $w^{\max(k,\ell)}(I)$. Since $w^{\max(k,\ell)}(I)$ is an ideal, there exists $z \in w^{\max(k,\ell)}(I) \subseteq w^\infty(I)$ such that $x \leq z$ and $y \leq z$. □

Recall that in the Karp-Miller algorithm for Petri nets, the level of an ideal remains unchanged when applying a transition, and increases when accelerated. This holds because the completion of a Petri net has strong-strict monotonicity. We introduce a more general (i.e. weaker) type of monotonicity that essentially yields the same behaviour.

Let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a WSTS. We define the *level* of an ideal $I \in \mathrm{Idl}(X)$ as follows. If $I \in A_n$ for some $n \in \mathbb{N}$, then $\mathrm{lvl}(I)$ is the smallest such n, and otherwise $\mathrm{lvl}(I) \stackrel{\text{def}}{=} \infty$. We say that the completion of $\mathcal{S}$ has *leveled-strong-strict monotonicity* if for every $I, I', J \in \mathrm{Idl}(X)$ and $w \in \Sigma^*$ such that $\mathrm{lvl}(I) \neq \infty$, the following holds:

$$\text{if } I \subset I', I \overset{w}{\rightsquigarrow} J \text{ and } \mathrm{lvl}(I) = \mathrm{lvl}(J), \text{ then } I' \overset{w}{\rightsquigarrow} J' \text{ for some } J' \in \mathrm{Idl}(X) \text{ s.t. } J \subset J'.$$

In other words, leveled-strong-strict monotonicity only requires strong-strict monotonicity to hold between ideals of the same level.

Petri nets and their completions enjoy strong-strict monotonicity (hence also leveled-strong-strict monotonicity), but strong-strict monotonicity is not inherited by the completion of some extensions such as post-self-modifying nets and ω-Petri nets.

Let us recall the model of post-self-modifying nets [Val78] for which there is a Karp-Miller algorithm. In post-self-modifying nets, transitions consume tokens as in Petri nets but they may add the result of applying a (different) positive affine function in each place. It has been shown [FMP04] that post-self-modifying nets are WSTS with strong-strict monotonicity on $\mathbb{N}^d$. Their completions are still WSTS with strong monotonicity on $\mathbb{N}_\omega^d$, but they are not strictly monotone on $\mathbb{N}_\omega^d$ (contrary to Figure 3 in [FMP04]). Let us show here that completions of post-self-modifying nets are not strictly monotone. Let us consider a post-self-modifying net N with two places p_1, p_2 and an unique transition t that adds the contents of p_1 onto p_2. Consider the two ω-markings $(\omega, 0) < (\omega, \omega)$ from $\mathbb{N}_\omega^2$ and the firing of transition t, extended on $\mathbb{N}_\omega^2$, from both ω-markings. We obtain $(\omega, 0) \xrightarrow{t} (\omega, \omega)$ and $(\omega, \omega) \xrightarrow{t} (\omega, \omega)$. Since $(\omega, \omega) \not< (\omega, \omega)$, transition t is not strictly increasing over $\mathbb{N}_\omega^2$, even if t is strictly increasing over $\mathbb{N}^2$. Hence the completion of N does not satisfy strict monotonicity.

Therefore, post-self-modifying nets are WSTS with strong-strict monotonicity and that their completions are WSTS with strong monotonicity. However, they are not *strictly* monotone.

Recall that ω-Petri nets are Petri nets with arcs labeled by coefficients from $\mathbb{N}_\omega$ instead of $\mathbb{N}$. The semantics remains the same for coefficients over $\mathbb{N}$. Every arc from a place p to a transition t, which is labeled by ω, consumes an arbitrary number of tokens from p when t is fired. Similarly, every arc from a transition t to a place p, which is labeled by ω, produces an arbitrary number of tokens in p when t is fired. In particular, in the completion of an

ω-Petri net, an arc from t to p labeled by ω increases the contents of p to ω whenever t is fired. See Figure 1 for an example of an ω-Petri net, and [GHPR15] for precise definitions.

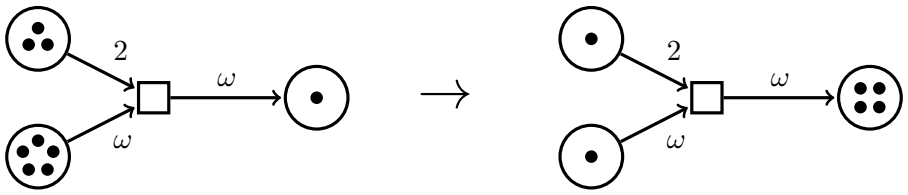

Figure 1: *Left*: example of an ω-Petri net marked (counterclockwise) with $(3, 5, 1)$. *Right*: example of a possible marking, i.e. $(1, 1, 4)$, obtained after firing the unique transition; the other possible markings are $(1, y, z)$ where $0 \leq y \leq 5$ and $z \geq 1$. Over the completion of the same ω-Petri net, the ideal $\downarrow 3 \times \downarrow 5 \times \downarrow 1$ leads to $\downarrow 1 \times \downarrow 5 \times \mathbb{N}$ when firing the unique transition; or equivalently $(3, 5, 1)$ leads to $(1, 5, \omega)$ in the ω-representation of the ideals.

It is known that ω-Petri nets are WSTS with strong-strict monotonicity and their completions are still WSTS with strong monotonicity [BFM18] but they are not *strictly* monotone. Indeed, consider the ω-Petri net with a single place p and a unique transition t with a single arc from t to p labeled by ω. We have $\downarrow 5 \xrightarrow{t} \mathbb{N}$, $\downarrow 6 \xrightarrow{t} \mathbb{N}$, $\downarrow 5 \subset \downarrow 6$, but not $\mathbb{N} \subset \mathbb{N}$. As a matter of fact, we will prove in Proposition 4.5 and Proposition 4.6 that post-self-modifying nets and ω-Petri nets have *leveled-strong-strict monotonicity*.

We may now show the following:

Proposition 3.11. *Let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a WSTS such that $\widehat{\mathcal{S}}$ is deterministic and has leveled-strong-strict monotonicity. Let $I \in \mathrm{Idl}(X)$ and $w \in \Sigma^+$ be such that $\mathrm{lvl}(I) \neq \infty$ and $Post_{\widehat{\mathcal{S}}}(I, w) \neq \emptyset$. The following holds:*

(1) $\mathrm{lvl}(w(I)) \geq \mathrm{lvl}(I)$,
(2) *if $w^\infty(I) \neq I$, then* $\mathrm{lvl}(w^\infty(I)) > \mathrm{lvl}(I)$,

Proof.

(1) We prove the claim by induction on $n = \mathrm{lvl}(I)$. If $n = 1$, then the claim trivially holds since $A_0 = \emptyset$ and hence $\mathrm{lvl}(J) \geq 1$ for every ideal J. Suppose that $n > 1$ and that the claim holds for levels smaller than n. Since $n > 1$ and n is the smallest index such that $I \in A_n$, there exists an acceleration candidate $I_0, I_1, \ldots$ below I such that $I_i \in A_{n-1}$ and $I_i \notin A_{n-2}$ for some $i \in \mathbb{N}$. In other words, $\mathrm{lvl}(I_i) = n - 1$.

Observe that $I_{i+1}, I_{i+2}, \ldots$ is also an acceleration candidate below I. Thus, there exists $j > i$ such that $I_j \in A_{n-1}$. By repeating this process, we obtain an acceleration candidate $I_{i_0}, I_{i_1}, \ldots$ below I such that $I_{i_j} \in A_{n-1}$ for every $j \in \mathbb{N}$. Thus, by $\mathrm{lvl}(I_0) = n - 1$ and by Proposition 3.7 (1), we have $\mathrm{lvl}(I_{i_0}) = \mathrm{lvl}(I_{i_1}) = \cdots = n - 1$. Hence, by leveled-strong-strict monotonicity and determinism of $\widehat{S}$, we have

$$w(I_{i_0}) \subset w(I_{i_1}) \subset \cdots \subseteq w(I). \tag{3.1}$$

Observe that (3.1) yields an acceleration candidate below $w(I)$. Thus, there exists $\ell \in \mathbb{N}$ such that $w(I_{i_\ell}) \in A_{\mathrm{lvl}(w(I))-1}$. Therefore, we are done since:

$$\begin{aligned}
\mathrm{lvl}(w(I)) &\geq \mathrm{lvl}(w(I_{i_\ell})) + 1 && (\text{by } w(I_{i_\ell}) \in A_{\mathrm{lvl}(w(I))-1}) \\
&\geq \mathrm{lvl}(I_{i_\ell}) + 1 && (\text{by ind. hyp. since } \mathrm{lvl}(I_{i_\ell}) < n) \\
&= (n-1) + 1 \\
&= n \\
&= \mathrm{lvl}(I).
\end{aligned}$$

(2) Assume $w^\infty(I) \neq I$. For the sake of contradiction, suppose that $\mathrm{lvl}(w^\infty(I)) \leq \mathrm{lvl}(I)$. Since $w^\infty(I) \neq I$, the sequence $I, w(I), w^2(I), \ldots$ is an acceleration candidate. By definition of $w^\infty(I)$, this acceleration candidate is below $w^\infty(I)$. Moreover, it goes through $A_{\mathrm{lvl}(I)-1}$, and hence there exists $k \in \mathbb{N}$ such that $\mathrm{lvl}(w^k(I)) = \mathrm{lvl}(I) - 1$. This contradicts (1). □

4. The Ideal Karp-Miller algorithm

We have now introduced all the concepts necessary to present our generalization of the Karp-Miller algorithm. This algorithm applies to a new class[2] of WSTS that enjoy all of the generalized properties of Petri nets:

Definition 4.1. A *very-WSTS* is a labeled WSTS $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ such that:

- $\mathcal{S}$ has strong monotonicity,
- $\widehat{\mathcal{S}}$ is a deterministic WSTS with leveled-strong-strict monotonicity,
- $\mathrm{Idl}(X)$ has finitely many levels.

Note that the completion $\widehat{\mathcal{S}}$ of a WSTS $\mathcal{S}$ always have strong monotonicity [BFM18]. However, it does not necessarily have leveled-strong-strict monotonicity. Moreover, if it does, it is not necessarily the case that $\mathcal{S}$ has strong monotonicity. In other words, although the first condition of Definition 4.1 may first appear redundant, it is not the case:

Proposition 4.2. *The first condition of Definition 4.1 is not redundant.*

Proof. We construct a WSTS $\mathcal{S}$ that satisfies all the requirements of a very-WSTS except for strong monotonicity. Let $\mathcal{S} = (\mathbb{N}, \xrightarrow{\{t\}}, \leq)$ be the ordered transition system such that $m \xrightarrow{t} m \div 2$ if m is even, and $m \xrightarrow{t} n$ for every $n \in \mathbb{N}$ otherwise. Since $\mathbb{N}$ is well-quasi-ordered, it suffices to show that $\mathcal{S}$ is monotone in order to show that it is a WSTS. Let $m, m', n \in \mathbb{N}$ be such that $m \xrightarrow{t} n$ and $m' \geq m$. If m' is odd, then $m' \xrightarrow{t} n$ and we are done. If m' is even, then it can be repeatedly halved until some odd number is obtained, after which we can reach n in one step, i.e. $m' \xrightarrow{*} n$. Observe that $\mathcal{S}$ is not strongly monotone since $1 \xrightarrow{t} 3$, but $3 \notin \downarrow 1 = \downarrow \mathrm{Post}_{\mathcal{S}}(2, t)$.

Let us now show that $\widehat{\mathcal{S}}$ is a deterministic WSTS with leveled-strong-strict monotonicity. It is readily seen that $\widehat{\mathcal{S}}$ is deterministic since:

$$\mathrm{Post}_{\widehat{\mathcal{S}}}(\downarrow 0, t) = \{\downarrow 0\} \text{ and } \mathrm{Post}_{\widehat{\mathcal{S}}}(I, t) = \{\mathbb{N}\} \text{ for every ideal } I \neq \downarrow 0.$$

[2]Note that the definition of very-WSTS given here is slightly more general than the one that appeared in the preliminary version of this paper [BFGL17]. More precisely, strong-strict monotonicity is replaced here with leveled-strong-strict monotonicity, which allows to encompass models such as ω-Petri nets.

It remains to show that $\widehat{\mathcal{S}}$ has monotonicity and leveled-strong-strict monotonicity. Let $I, I', J \in \mathrm{Idl}(\mathbb{N})$ be such that $I \overset{t}{\rightsquigarrow} J$ and $I \subset I'$. Note that $I' \supset I \supseteq {\downarrow}0$, hence I' must contain at least one odd number. Therefore, $I' \overset{t}{\rightsquigarrow} \mathbb{N}$ which shows (standard) monotonicity. Since $I \subset I' \subseteq \mathbb{N}$, we have $I \neq \mathbb{N}$. Thus, if $\mathrm{lvl}(I) = \mathrm{lvl}(J)$, then we have $J \neq \mathbb{N}$. Since $I' \overset{t}{\rightsquigarrow} \mathbb{N}$ and $\mathbb{N} \supset J$, this shows leveled-strong-strict monotonicity. □

We claim that the class of very-WSTS includes Petri nets (and hence vector addition systems with/without states), ω-Petri nets [GHPR15], post-self-modifying nets [Val78] and strongly increasing ω-recursive nets [FMP04] for which Karp-Miller algorithms were known.

Recall that a *strongly increasing function* $f : \mathbb{N}^d \to \mathbb{N}^d$ is a nondecreasing function defined on an upward closed set of $\mathbb{N}^d$ that satisfies the following strongly increasing property:

$$\text{for every } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^d, \text{for every } P \subseteq \{1, 2, \ldots, d\},\ \boldsymbol{x} \leq_P \boldsymbol{y} \implies f(\boldsymbol{x}) \leq_P f(\boldsymbol{y}),$$

where the ordering $\leq_P$ is defined by

$$\boldsymbol{x} \leq_P \boldsymbol{y} \overset{\text{def}}{\iff} \boldsymbol{x} \leq \boldsymbol{y} \wedge \boldsymbol{x}(i) < \boldsymbol{y}(i) \text{ for every } i \in P.$$

A *strongly increasing recursive net* N is a finite set of strongly increasing recursive functions. A *strongly increasing ω-recursive net* N is a strongly increasing recursive net such that the continuous extensions of the functions $f : \mathbb{N}^d_\omega \to \mathbb{N}^d_\omega$ satisfy the previous strongly increasing property but over $\mathbb{N}^d_\omega$ instead (see, e.g., [FG12] for a definition of continuous extension). Let us write $\mathcal{S}_N$ for the transition system naturally associated with a net N. We may observe that $\mathcal{S}_{\widehat{N}} = \widehat{\mathcal{S}}_N$.

Since nondecreasing functions of post-self-modifying nets and of strongly increasing ω-recursive nets are incomparable, we define another class of nondecreasing functions that subsumes the two previous ones. Let us identify the nondecreasing functions over $\mathbb{N}^d$ that are strictly increasing, but only on the subset of $\mathbb{N}^d$ such that $\boldsymbol{x}$ and $f(\boldsymbol{x})$ have the same number of ω's.

Definition 4.3. A *leveled-increasing* partial function $f \colon \mathbb{N}^d \to \mathbb{N}^d$ is a nondecreasing partial function such that its continuous extension $\widehat{f} \colon \mathbb{N}^d_\omega \to \mathbb{N}^d_\omega$ satisfies the following property: for every $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{N}^d_\omega$ such that $\boldsymbol{x}$ and $f(\boldsymbol{x})$ contain the same number of ω (in terms of ideals, $\boldsymbol{x}$ and $f(\boldsymbol{x})$ have the same level), the following holds:

$$\boldsymbol{x} < \boldsymbol{x}' \implies f(\boldsymbol{x}) < f(\boldsymbol{x}').$$

A *leveled-increasing recursive net* is a finite set of leveled-increasing recursive partial functions.

Let us remark that the composition of two leveled-increasing partial functions is still a leveled-increasing partial function, hence the associated transition system is a WSTS with leveled-strong-strict monotonicity.

Proposition 4.4. *Leveled-increasing recursive nets are very-WSTS.*

Proof. Let N be a leveled-increasing recursive net. By hypothesis, $\mathcal{S}_N$ has strong monotonicity since the partial functions of N are nondecreasing. Moreover, $\mathcal{S}_N$ can be shown to be a deterministic WSTS, and $\mathcal{S}_{\widehat{N}}$ is a deterministic WSTS with leveled-strong-strict monotonicity because finite composition of partial functions in N is leveled-increasing. Finally, the set of states is $\mathbb{N}^d$ and we know that $\mathrm{Idl}(\mathbb{N}^d)$ has finitely many levels. Therefore, $\mathcal{S}_N$ is a very-WSTS. □

Since Petri nets (or vector addition systems with/without states) and strongly increasing ω-recursive nets are leveled-increasing recursive nets by definition, to prove our claim it is sufficient to prove that post-self-modifying nets and ω-Petri nets are leveled increasing recursive nets.

Proposition 4.5. *Post-self-modifying nets are very-WSTS.*

Proof. Let N be a post-self-modifying net. Let us prove that each partial function f occurring on a transition t of N is leveled-increasing. Recall that $f(\boldsymbol{x}) = \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{b}$ where $\boldsymbol{A}$ is greater or equal to the identity matrix componentwise, and $\boldsymbol{b} \in \mathbb{Z}^d$.

Recall that we want to show that for every $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{N}_\omega^d$ such that $\boldsymbol{x}$ and $f(\boldsymbol{x})$ contain the same number of ω's, the following holds:

$$\boldsymbol{x} < \boldsymbol{x}' \implies f(\boldsymbol{x}) < f(\boldsymbol{x}').$$

Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{N}_\omega^d$ be such that $\boldsymbol{x}$ and $f(\boldsymbol{x})$ contain the same number of ω's and such that $\boldsymbol{x} < \boldsymbol{x}'$. Since $\boldsymbol{A} \geq \boldsymbol{0}$, we have $f(\boldsymbol{x}) \leq f(\boldsymbol{x}')$. Moreover, there exists a least index $1 \leq \ell \leq d$ such that $\boldsymbol{x}(\ell) < \boldsymbol{x}'(\ell)$, and in particular $\boldsymbol{x}(\ell) \neq \omega$. Since $\boldsymbol{A}$ is greater or equal to the identity matrix, then $\boldsymbol{y}(i) = \omega$ implies $f(\boldsymbol{y})(i) = \omega$ for every $\boldsymbol{y}$, i.e. ω's cannot disappear. Since, by hypothesis, the number of ω's is the same in $\boldsymbol{x}$ and $f(\boldsymbol{x})$, this means that f does not create new ω's in a new position and hence $\boldsymbol{x}$ and $f(\boldsymbol{x})$ have exactly the same ω's in the same positions. Thus, $\boldsymbol{x}(\ell) \neq \omega$ and $f(\boldsymbol{x})(\ell) \neq \omega$.

Since $\boldsymbol{A}(\ell, \ell) \geq 1$ and $\boldsymbol{x}'(\ell) > \boldsymbol{x}(\ell)$, we deduce that

$$\boldsymbol{A}(\ell, \ell) \cdot \boldsymbol{x}'(\ell) > A(\ell, \ell) \cdot \boldsymbol{x}(\ell) \neq \omega.$$

Moreover, for every $1 \leq j \leq d$:

$$A(\ell, j) \cdot \boldsymbol{x}'(j) \geq A(\ell, j) \cdot \boldsymbol{x}(j) \neq \omega \text{ since } f(\boldsymbol{x})(\ell) \neq \omega.$$

Thus, $f(\boldsymbol{x}')(\ell) = \sum_{j=1}^{d} A(\ell, j) \cdot \boldsymbol{x}'(j) > f(\boldsymbol{x})(\ell)$. Therefore, we conclude that $f(\boldsymbol{x}') > f(\boldsymbol{x})$ and hence that f is leveled-increasing. Hence, from Proposition 4.4, we deduce that N is a very-WSTS. □

Proposition 4.6. *ω-Petri nets are very-WSTS.*

Proof. Let N be a ω-Petri net with places P and transitions T. Let $t \in T$ be a transition of N. Let $f \colon \mathbb{N}^P \to \mathbb{N}^P$ be the partial function such that $f(\boldsymbol{x})$ is the marking obtained by firing t in $\boldsymbol{x}$, provided that t is enabled in $\boldsymbol{x}$. Let us prove that f is leveled-increasing. Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{N}_\omega^P$ be such that $\boldsymbol{x} < \boldsymbol{x}'$ and

$$f(\boldsymbol{x}) \text{ contains the same number of } \omega\text{'s as } \boldsymbol{x} \tag{4.1}$$

We need to show that $f(\boldsymbol{x}) < f(\boldsymbol{x}')$.

Let $p \in P$. Note that $\boldsymbol{x}(p) = \omega$ implies $f(\boldsymbol{x})(p) = \omega$, i.e. t cannot remove an ω in $\widehat{S}$. Let $\mathrm{Pre}(p, t)$ and $\mathrm{Post}(p, t)$ denote respectively the number of tokens consumed and produced by t in p. Let $\sim$ stand for $\leq$ or $<$ depending on whether $\boldsymbol{x}(p) \leq \boldsymbol{x}'(p)$ or $\boldsymbol{x}(p) < \boldsymbol{x}'(p)$. It suffices to show that $f(\boldsymbol{x})(p) \sim f(\boldsymbol{x}')(p)$. We make a case distinction on whether $\boldsymbol{x}(p)$ equals ω or not.

Case "$\boldsymbol{x}(p) \neq \omega$": Let

$$a \stackrel{\text{def}}{=} \begin{cases} \mathrm{Pre}(p, t) & \text{if } \mathrm{Pre}(p, t) \neq \omega, \\ 0 & \text{otherwise.} \end{cases} \qquad b \stackrel{\text{def}}{=} \mathrm{Post}(p, t).$$

We must have $b \neq \omega$ since we would otherwise have $f(\boldsymbol{x})(p) = \omega$ which would contradict (4.1). Thus, $f(\boldsymbol{x})(p) = \boldsymbol{x}(p) - a + b \sim \boldsymbol{x}'(p) - a + b = f(\boldsymbol{x}')(p)$.

Case "$\boldsymbol{x}(p) = \omega$": Since $\boldsymbol{x}'(p) \geq \boldsymbol{x}(p) = \omega$, we have $\boldsymbol{x}'(p) = \omega$. Therefore, $f(\boldsymbol{x})(p) = \omega = f(\boldsymbol{x}')(p)$ and we are done. Hence, from Proposition 4.4, we deduce that N is a very-WSTS. □

By the two previous propositions, we obtain the following:

Corollary 4.7. *Petri nets (or vector addition systems with/without states), ω-Petri nets, post-self-modifying nets and strongly increasing ω-recursive nets are very-WSTS.*

Note that very-WSTS do not include transfer Petri nets, since $\widehat{\mathcal{S}}$ does not have leveled-strong-strict monotonicity, and unordered data Petri nets, since $\mathrm{Idl}(X)$ has infinitely many levels. Observe that $\widehat{\mathcal{S}}$ may be deterministic (and finitely branching) even when $\mathcal{S}$ is not, and even when $\mathcal{S}$ is not finitely branching, as the example of ω-Petri nets shows.

Algorithm 4.1: Ideal Karp-Miller algorithm.

```
1  initialize a tree 𝒯 with root r: I_0
2  while 𝒯 contains an unmarked node c: I do
3      if c has an ancestor c': I' s.t. I' = I then
4          mark c                                   /* stop exploration */
5      else
6          if c has an ancestor c': I' such that I' ⊂ I then
7              let c' be the closest such ancestor
8              w ← sequence of labels from c' to c
9              if w^∞(I) ≠ I then
10                 replace c: I by c: w^∞(I)          /* accelerate */
11         for a ∈ Σ do
12             if Post_Ŝ(I, a) ≠ ∅ then
13                 add arc labeled by a from c to a new child d: a(I)
14         mark c
15 return 𝒯
```

We now present the *Ideal Karp-Miller algorithm (IKM)*[3] for very-WSTS in Algorithm 4.1. The algorithm starts from an ideal I_0, successively computes its successors in $\widehat{\mathcal{S}}$ and performs accelerations as in the classical Karp-Miller algorithm for Petri nets. For every node $c\colon I$ of the tree built by the algorithm, let $\mathrm{ideal}(c) \stackrel{\text{def}}{=} I$ and $\mathrm{lvl}(c) \stackrel{\text{def}}{=} \mathrm{lvl}(I)$. Let us first show that the algorithm terminates.

Theorem 4.8. *Algorithm 4.1 terminates for very-WSTS.*

Proof. Since $\mathrm{Idl}(X)$ has finitely many levels, $\mathrm{lvl}(I) \neq \infty$ for every $I \in \mathrm{Idl}(X)$. Moreover,

$$\mathrm{lvl}(c) \text{ is non-decreasing on each branch of } \mathcal{T}, \tag{4.2}$$

[3]Note that the algorithm given here is slightly more general and simplified than the one that appeared in the preliminary version of this paper [BFGL17]. Here we allow for some nested accelerations while this was explicitly disallowed in the algorithm of [BFGL17].

that is: for every branch $c_0, c_1, \dots$ of $\mathcal{T}$, we have $\mathrm{lvl}(c_0) \leq \mathrm{lvl}(c_1) \leq \cdots$. This observation follows from Proposition 3.11 combined with the fact that the algorithm constructs a node's ideal either from applying a transition to its parent's ideal, or by performing an acceleration.

The rest of the argument is as for the classical Karp-Miller algorithm. Suppose the algorithm does not terminate. Let $\mathcal{T}_n$ be the finite tree obtained after n iterations. The infinite sequence $\mathcal{T}_0, \mathcal{T}_1, \dots$ defines a unique infinite tree $\mathcal{T}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{T}_n$. Since $\widehat{\mathcal{S}}$ is finitely branching, $\mathcal{T}_\infty$ is also finitely branching. Therefore, $\mathcal{T}_\infty$ contains an infinite path $c_0, c_1, \dots$ by König's lemma. By (4.2), and since $\mathrm{Idl}(X)$ has finitely many levels, there exists $k, m \in \mathbb{N}$ such that

$$\mathrm{lvl}(c_k) = \mathrm{lvl}(c_{k+1}) = \cdots = m. \tag{4.3}$$

Since $\widehat{\mathcal{S}}$ is a WSTS, the set $\mathrm{Idl}(X)$ is well-quasi-ordered, and hence we can find two indices i, j such that $k \leq i < j$ and $\mathrm{ideal}(c_i) \subseteq \mathrm{ideal}(c_j)$. If $\mathrm{ideal}(c_i) = \mathrm{ideal}(c_j)$, then line 3 of the algorithm would have stopped the exploration of the path. Therefore, $\mathrm{ideal}(c_i) \subset \mathrm{ideal}(c_j)$. Let $\ell \in \mathbb{N}$ be the largest index such that $\ell < j$ and $\mathrm{ideal}(c_\ell) \subset \mathrm{ideal}(c_j)$. Let $w \in \Sigma^+$ be the sequence of labels from c_ℓ to c_j. Note that $k \leq i \leq \ell < j$. Therefore, by (4.3) and Proposition 3.11, no acceleration occurred between c_ℓ and c_j, and hence

$$\mathrm{ideal}(c_\ell) \overset{w}{\rightsquigarrow} \mathrm{ideal}(c_j).$$

Let $I \overset{\text{def}}{=} \mathrm{ideal}(c_\ell)$ and $J = \mathrm{ideal}(c_j)$. By (4.3), $\mathrm{lvl}(I) = \mathrm{lvl}(J)$. Therefore, by leveled-strong-strict monotonicity of $\widehat{S}$, the sequence $J, w(J), w^2(J), \dots$ is an acceleration candidate, and hence $w^\infty(J) \neq J$. Thus, line 10 has been executed on c_j, which implies that $\mathrm{lvl}(I) < \mathrm{lvl}(J)$ by Proposition 3.11. This contradicts (4.3), which completes the proof. □

4.1. **Properties of the algorithm.** Let $\mathcal{T}_I$ denote the tree returned by Algorithm 4.1 on input $(\mathcal{S}, I)$. Let $D_I \overset{\text{def}}{=} \bigcup_{c\in\mathcal{T}_I} \mathrm{ideal}(c)$. We claim that $D_I = {\downarrow}\mathrm{Post}^*_{\mathcal{S}}(I)$. Instead of proving this claim directly, we take traces into consideration and prove a stronger statement. We define two word automata that will be useful for this purpose.

Definition 4.9. The *stuttering automaton*[4] is the finite word automaton $\mathcal{A}_I$ obtained by making all of the states of $\mathcal{T}_I$ accepting, by taking the root r as the initial state, and by taking the arcs of $\mathcal{T}_I$ as transitions, together with the following additional transitions:

- If a leaf c of $\mathcal{T}_I$ has an ancestor c' such that $\mathrm{ideal}(c) = \mathrm{ideal}(c')$, then a transition from c to c' labeled by ε is added to $\mathcal{A}_I$.

The *Karp-Miller automaton* is the automaton $\mathcal{K}_I$ obtained by extending $\mathcal{A}_I$ as follows:

- If a node c of $\mathcal{T}_I$ has been accelerated because of an ancestor c', then a transition from c to c' labeled by ε is added to $\mathcal{K}_I$.

Both $\mathcal{A}_I$ and $\mathcal{K}_I$ can be computed from $\mathcal{T}_I$. Moreover, they give precious information about the traces of $\mathcal{S}$. Let $\mathrm{L}(\mathcal{A}_I)$ and $\mathrm{L}(\mathcal{K}_I)$ denote the language over Σ accepted by $\mathcal{A}_I$ and $\mathcal{K}_I$. Recall that $\preceq$ denotes the subword ordering. We will show the following theorem:

Theorem 4.10. *For every very-WSTS* $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ *and* $I \in \mathrm{Idl}(X)$,

$$D_I = {\downarrow}\mathit{Post}^*_{\mathcal{S}}(I),\ \mathrm{Traces}_{\mathcal{S}}(I) \subseteq \mathrm{L}(\mathcal{A}_I)\ \textit{and}\ \mathrm{L}(\mathcal{K}_I) \subseteq {\downarrow_{\preceq}}\mathrm{Traces}_{\mathcal{S}}(I).$$

[4] We use the term *stuttering* as paths of the automaton correspond to stuttering paths of [GHPR15].

In particular, for every $x \in X$, $D_{\downarrow x} = \downarrow Post^*_{\mathcal{S}}(x)$, $\downarrow_{\preceq} \mathrm{L}(\mathcal{K}_{\downarrow x}) = \downarrow_{\preceq} \mathrm{Traces}_{\mathcal{S}}(x)$, *and* $\downarrow_{\preceq} \mathrm{Traces}_{\mathcal{S}}(x)$ *is a computable regular language.*

The proof of Theorem 4.10 follows from the forthcoming Propositions 4.11 and 4.12 describing the relations between traces of $\mathcal{A}_I$ and $\mathcal{K}_I$ with traces of $\mathcal{S}$ and $\widehat{\mathcal{S}}$. We write $c \overset{w}{\dashrightarrow}_{\mathcal{T}} c'$, $c \overset{w}{\dashrightarrow}_{\mathcal{A}} c'$ and $c \overset{w}{\dashrightarrow}_{\mathcal{K}} c'$ whenever node c' can be reached by reading w from c in $\mathcal{T}_I$, $\mathcal{A}_I$ and $\mathcal{K}_I$ respectively.

Proposition 4.11. *Let* $\mathcal{S} = (X, \overset{\Sigma}{\to}, \leq)$ *be a very-WSTS and let* $I_0 \in \mathrm{Idl}(X)$. *For every* $y, z \in X$, $w \in \Sigma^*$ *and* $c \in \mathcal{A}_{I_0}$, *if* $y \overset{w}{\to} z$ *and* $y \in \mathrm{ideal}(c)$, *then there exists* $d \in \mathcal{A}_{I_0}$ *such that* $c \overset{w}{\dashrightarrow}_{\mathcal{A}} d$ *and* $z \in \mathrm{ideal}(d)$.

Proof. The proof is by induction on $|w|$. If $|w| = 0$, then $w = \varepsilon$, which implies $z = y$. Thus, it suffices to take $d \overset{\text{def}}{=} c$.

Assume $|w| > 0$ and that the claims holds for words of length less than $|w|$. There exist $u \in \Sigma^*$, $a \in \Sigma$ and $y' \in X$ such that $w = ua$ and $y \overset{u}{\to} y' \overset{a}{\to} z$. By induction hypothesis, there exists a node $c' \in \mathcal{A}_{I_0}$ such that $c \overset{u}{\dashrightarrow}_{\mathcal{A}} c'$ and $y' \in \mathrm{ideal}(c')$. Let $I \overset{\text{def}}{=} \mathrm{ideal}(c')$. Since $y' \overset{a}{\to} z$ and $y' \in I$, there exists some $J \in \mathrm{Idl}(X)$ such that $z \in J$ and $I \overset{a}{\rightsquigarrow} J$. If c' has a successor under a labeled by J, then we are done. Otherwise, there are two cases to consider.

- If c' has no successor under a, then c' must be a leaf of T_{I_0}. Thus, c' has an ancestor c'' in T_{I_0} such that $\mathrm{ideal}(c') = \mathrm{ideal}(c'')$. Thus, $c' \overset{\varepsilon}{\dashrightarrow}_{\mathcal{A}} c''$. Now, c'' has a successor d under a, otherwise it would also be a leaf of T_{I_0}, which is impossible. Therefore, $J = \mathrm{ideal}(d)$, and hence $c \overset{u}{\dashrightarrow}_{\mathcal{A}} c' \overset{\varepsilon}{\dashrightarrow}_{\mathcal{A}} c'' \overset{a}{\dashrightarrow}_{\mathcal{A}} d$ and $z \in \mathrm{ideal}(d)$.
- If c has a successor d under a, then J has been accelerated. Therefore, $\mathrm{ideal}(d) = v^\infty(J)$ for some $v \in \Sigma^+$. By definition of accelerations, $J \subseteq v^\infty(J)$. Therefore, $c \overset{u}{\dashrightarrow}_{\mathcal{A}} c' \overset{a}{\dashrightarrow}_{\mathcal{A}} d$ and $y \in \mathrm{ideal}(d)$. □

Proposition 4.12. *Let* $\mathcal{S} = (X, \overset{\Sigma}{\to}, \leq)$ *be a very-WSTS and let* $I_0 \in \mathrm{Idl}(X)$. *For every* $z \in X$, $w \in \Sigma^*$ *and* $c, d \in \mathcal{K}_{I_0}$, *if* $c \overset{w}{\dashrightarrow}_{\mathcal{K}} d$ *and* $z \in \mathrm{ideal}(d)$, *then there exist* $y \in \mathrm{ideal}(c)$, $w' \succeq w$ *and* $z' \geq z$ *such that* $y \overset{w'}{\longrightarrow} z'$.

Proof. The proof is by induction on $|w|$. If $|w| = 0$, then $w = \varepsilon$. We stress the fact that even though w is empty, d might differ from c since $\mathcal{K}_{I_0}$ contains ε-transitions. However, by definition of $\mathcal{K}_{I_0}$, we know that $\mathrm{ideal}(d) \subseteq \mathrm{ideal}(c)$. Therefore, $z \in \mathrm{ideal}(c)$, and we are done since $z \overset{\varepsilon}{\to} z$.

Suppose that $|w| > 0$. Assume the claim holds for every word of length less than $|w|$. There exist $u, v \in \Sigma^*$, $a \in \Sigma$ and $d' \in \mathcal{K}_{I_0}$ such that $w = uav$, $c \overset{u}{\dashrightarrow}_{\mathcal{K}} d' \overset{a}{\dashrightarrow}_{\mathcal{K}} d \overset{v}{\dashrightarrow}_{\mathcal{K}} d$ and d' is the parent of d in T_{I_0}. Let $I \overset{\text{def}}{=} \mathrm{ideal}(c)$, $J \overset{\text{def}}{=} \mathrm{ideal}(d')$, $K \overset{\text{def}}{=} \mathrm{ideal}(d)$, and $K' \overset{\text{def}}{=} a(J)$. By induction hypothesis, there exist $y_K \in K$, $v' \succeq v$ and $z' \geq z$ such that $y_K \overset{v'}{\longrightarrow} z'$.

- If $K = K'$, then $J \overset{a}{\rightsquigarrow} K$. By definition of $\overset{a}{\rightsquigarrow}$, there exist $y_J \in J$ and $y'_K \geq y_K$ such that $y_J \overset{a}{\to} y'_K$. By induction hypothesis, there exist $y_I \in I$, $u' \succeq u$ and $y'_J \geq y_J$ such that $y_I \overset{u'}{\longrightarrow} y'_J$. By strong monotonicity of $\mathcal{S}$, there exists $z'' \geq z'$ such that $y_I \overset{u'av'}{\longrightarrow} z''$. We are done since $u'av' \succeq uav$.

- If $K \neq K'$, then K was obtained through an acceleration. Therefore, $K = \sigma^\infty(K')$ for some $\sigma \in \Sigma^+$. This implies that $y_K \in \sigma^k(K')$ for some $k \in \mathbb{N}$. Let $L \stackrel{\text{def}}{=} \sigma^k(K')$. Note that $J \overset{a}{\rightsquigarrow} K' \overset{\sigma^k}{\rightsquigarrow} L$. By Proposition 3.4(2), there exist $y_J \in J$ and $y'_K \geq y_K$ such that $y_J \xrightarrow{a\sigma^k} y'_K$. By induction hypothesis, there exist $y_I \in I$, $u' \succeq u$ and $y'_J \geq y_J$ such that $y_I \xrightarrow{u'} y'_J$. By strong monotonicity of $\mathcal{S}$, there exists $z'' \geq z'$ such that $y_I \xrightarrow{u'a\sigma^k v'} z''$. □

We may now prove the main theorem of this section:

Proof of Theorem 4.10.

(1) $\subseteq$: Let $y \in D_I$. There exist $w \in \Sigma^*$ and $c \in \mathcal{K}_I$ such that $r \overset{w}{\dashrightarrow}_{\mathcal{K}} c$ and $y \in \text{ideal}(c)$. By Proposition 4.12, there exist $x \in I$, $w' \succeq w$ and $y' \geq y$ such that $x \xrightarrow{w'} y$. Hence, $y \in \text{Post}^*_{\mathcal{S}}(x) \subseteq \text{Post}^*_{\mathcal{S}}(I_0) \subseteq {\downarrow}\,\text{Post}^*_{\mathcal{S}}(I)$.

$\supseteq$: Let $y \in {\downarrow}\,\text{Post}^*_{\mathcal{S}}(I)$. There exist $x \in I$, $w \in \Sigma^*$ and $y' \geq y$ such that $x \xrightarrow{w} y'$. By Proposition 4.11, there exists a node $c \in \mathcal{A}_I$ such that $r \overset{w}{\dashrightarrow}_{\mathcal{A}} c$ and $y' \in \text{ideal}(c)$. Since ideals are downward closed, $y \in \text{ideal}(c)$ which implies that $y \in D_I$.

(2) Let $w \in \text{Traces}_{\mathcal{S}}(I)$. There exist $x \in I$ and $y \in X$ such that $x \xrightarrow{w} y$. By Proposition 4.11, there exists a node $c \in \mathcal{A}_I$ such that $r \overset{w}{\dashrightarrow}_{\mathcal{A}} c$ and $y \in \text{ideal}(c)$. Therefore, $w \in \text{L}(\mathcal{A}_I)$.

(3) Let $w \in \text{L}(\mathcal{K}_I)$. There exists a node $c \in \mathcal{K}_I$ such that $r \overset{w}{\dashrightarrow}_{\mathcal{K}} c$. Let $y \in \text{ideal}(c)$. By Proposition 4.12, there exists $x \in I$, $w' \succeq w$ and $y' \geq y$ such that $x \xrightarrow{w'} y'$. Therefore, $w \in {\downarrow_\preceq}\,\text{Traces}_{\mathcal{S}}(I)$ since $w \preceq w'$. □

Corollary 4.13. *For every very-WSTS $\mathcal{S} = (X, \overset{\Sigma}{\to}, \leq)$ and every state $x \in X$,*

$$D_{{\downarrow}x} = {\downarrow}\,\textit{Post}^*_{\mathcal{S}}(x) \text{ and } {\downarrow_\preceq}\,\text{L}(\mathcal{K}_{{\downarrow}x}) = {\downarrow_\preceq}\,\text{Traces}_{\mathcal{S}}(x).$$

In particular, ${\downarrow_\preceq}\,\text{Traces}_{\mathcal{S}}(x)$ is a regular language computable from $\mathcal{S}$ and x.

Proof.

- By Theorem 4.10, we have $D_{{\downarrow}x} = {\downarrow}\,\text{Post}^*_{\mathcal{S}}({\downarrow}x)$. Moreover, by strong monotonicity of $\mathcal{S}$, we have ${\downarrow}\,\text{Post}^*_{\mathcal{S}}({\downarrow}x) = {\downarrow}\,\text{Post}^*_{\mathcal{S}}(x)$.
- By Theorem 4.10, we have

$$\text{Traces}_{\mathcal{S}}({\downarrow}x) \subseteq \text{L}(\mathcal{A}_{{\downarrow}x}) \subseteq \text{L}(\mathcal{K}_{{\downarrow}x}) \subseteq {\downarrow_\preceq}\,\text{Traces}_{\mathcal{S}}({\downarrow}x).$$

Therefore ${\downarrow_\preceq}\,\text{Traces}_{\mathcal{S}}({\downarrow}x) = {\downarrow_\preceq}\,\text{L}(\mathcal{A}_{{\downarrow}x}) = {\downarrow_\preceq}\,\text{L}(\mathcal{K}_{{\downarrow}x})$. Moreover, by strong monotonicity of $\mathcal{S}$, we have $\text{Traces}_{\mathcal{S}}({\downarrow}x) = \text{Traces}_{\mathcal{S}}(x)$. □

4.2. **Effectiveness of the algorithm.** The Ideal Karp-Miller algorithm can be implemented provided that

(1) ideals can be effectively manipulated, i.e., the set of encodings of $\text{Idl}(X)$ is recursive and the encoding of ${\downarrow}x$ is computable from $x \in X$ (see [BFM17] for a formal treatment of encodings),
(2) inclusion of ideals can be tested,
(3) $\text{Post}_{\widehat{\mathcal{S}}}(I, a)$ can be computed for every ideal I and $a \in \Sigma$, and
(4) $w^\infty(I)$ can be computed for every ideal I and sequence $w \in \Sigma^+$.

A class of WSTS satisfying (1–3) is called *completion-post-effective*, and a class satisfying (4) is called ∞-*completion-effective*. By Theorem 4.10, we obtain the following:

Theorem 4.14. *Let $\mathcal{C}$ be a completion-post-effective and ∞-completion-effective class of very-WSTS. The ideal decomposition of $\downarrow Post^*_{\mathcal{S}}(x)$ can be computed for every $\mathcal{S} = (X, \to, \leq) \in C$ and $x \in X$. In particular, coverability for $\mathcal{C}$ is decidable.*

5. Model checking liveness properties for very-WSTS

In this section, we show how the Ideal Karp-Miller algorithm can be used to test whether a very-WSTS violates a liveness property specified by an LTL formula. We follow classical constructions that have also been adapted to WSTS by Emerson and Namjoshi [EN98] without effectiveness constraints. Testing that $\mathcal{S}$ violates a property φ amounts to constructing a Büchi automaton $\mathcal{B}_{\neg\varphi}$ for $\neg\varphi$ and testing whether $\mathcal{B}_{\neg\varphi}$ accepts an infinite trace of $\mathcal{S}$. We first show that repeated coverability is decidable for very-WSTS under some effectiveness hypotheses. Then, we show how LTL model checking reduces to repeated coverability.

5.1. Deciding repeated coverability.

Let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a WSTS, let $x \in X$ and let $I \in \mathrm{Idl}(X)$. We say that $w \in \Sigma^*$ is (I, x)-*increasing* if there exist $y \in I$ and $z \in X$ such that $y \xrightarrow{w} z$ and $x \leq y \leq z$. We establish a necessary and sufficient condition for repeated coverability in terms of the stuttering automaton and (I, x)-increasing sequences:

Proposition 5.1. *Let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a very-WSTS and let $x, y \in X$. State y is repeatedly coverable from x if and only if there exist a state c of the stuttering automaton $\mathcal{A}_{\downarrow x}$, and a sequence $w \in \Sigma^+$, such that $c \xdashrightarrow{w}_{\mathcal{A}} c$ and w is* $(\mathrm{ideal}(c), y)$*-increasing.*

Proof. ($\Rightarrow$) Assume y is repeatedly coverable from x. There exist $y_0, y_1, \cdots \in X$, $v_0 \in \Sigma^*$ and $v_1, v_2, \ldots \in \Sigma^+$ such that

$$x \xrightarrow{v_0} y_0 \xrightarrow{v_1} y_1 \xrightarrow{v_2} \cdots$$

and $y_i \geq y$ for every $i \in \mathbb{N}$. By Proposition 4.11, there exist $c_0, c_1, \ldots \in \mathcal{A}_{\downarrow x}$ such that

$$r \xdashrightarrow{v_0}_{\mathcal{A}} c_0 \xdashrightarrow{v_1}_{\mathcal{A}} c_1 \xdashrightarrow{v_2}_{\mathcal{A}} \cdots$$

and $y_i \in \mathrm{ideal}(c_i)$ for every $i \in \mathbb{N}$. Since $\mathcal{A}_{\downarrow x}$ is finite, there exists $c \in \mathcal{A}_{\downarrow x}$ such that $I = \{i \in \mathbb{N} : c_i = c\}$ is infinite. Since X is well-quasi-ordered, there exist $i, j \in I$ such that $i < j$ and $y_i \leq y_j$. Let $w \stackrel{\text{def}}{=} v_{i+1} \cdots v_j$. We have $c \xdashrightarrow{w}_{\mathcal{A}} c$ and $|w| > 0$. Moreover, w is $(\mathrm{ideal}(c), y)$-increasing since $y_i \in \mathrm{ideal}(c)$, $y_i \xrightarrow{w} y_j$ and $y \leq y_i \leq y_j$.

($\Leftarrow$) Let $c \in \mathcal{A}_{\downarrow x}$ and $w \in \Sigma^+$ be such that $c \xdashrightarrow{w}_{\mathcal{A}} c$ and w is $(\mathrm{ideal}(c), y)$-increasing. Since w is $(\mathrm{ideal}(c), y)$-increasing, there exist $y' \in \mathrm{ideal}(c)$ and $y'' \in X$ such that $y \leq y' \leq y''$ and

$$y' \xrightarrow{w} y''. \tag{5.1}$$

Let $u \in \Sigma^*$ be the (unique) path from r to c in $\mathcal{T}_{\downarrow x}$. By Proposition 4.12, there exist $x' \in \mathrm{ideal}(r)$, $u' \succeq u$ and $z \geq y'$ such that

$$x' \xrightarrow{u'} z. \tag{5.2}$$

Since $\text{ideal}(r) = {\downarrow}x$, we have $x' \leq x$. By (5.2) and strong monotonicity of $\mathcal{S}$, $x \xrightarrow{u'} z'$ for some $z' \geq z$. Let $y_0 \stackrel{\text{def}}{=} z'$. By (5.1), $y_0 \geq y'$ and strong monotonicity of $\mathcal{S}$, we have $y_0 \xrightarrow{w} y_1$ for some $y_1 \geq y''$. Note that $y_1 \geq y'' \geq y'$, and hence again by strong monotonicity, $y_1 \xrightarrow{w} y_2$ for some $y_2 \geq y''$. By such successive application of strong monotonicity, we obtain $y_1, y_2, y_3, \ldots \in X$ such that $y_i \xrightarrow{w} y_{i+1}$ and $y_{i+1} \geq y''$ for every $i \in \mathbb{N}$. Therefore,

$$x \xrightarrow{u'} y_0 \xrightarrow{w} y_1 \xrightarrow{w} \cdots$$

and we are done since $y_0 = z' \geq z \geq y' \geq y$ and $y_i \geq y'' \geq y' \geq y$ for every $i \in \mathbb{N}$. □

Proposition 5.1 allows us to show the decidability of repeated coverability under the following effectiveness hypothesis. A class $\mathcal{C}$ of WSTS is *ideal-increasing-effective* if there is an algorithm that decides the following:

INPUT: $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq) \in \mathcal{C}$, $I \in \text{Idl}(X)$, $x \in I$ and a finite automaton A such that $\text{Post}_{\mathcal{S}}(I, w) \neq \emptyset$ for every $w \in L(A)$.

DECIDE: does there exist $w \in L(A)$ such that w is (I, x)-increasing?

Before proving decidability of repeated coverability, we first prove two useful observations on the stuttering automaton. For every node c of an IKM tree, we define num-accel(c) as the number of accelerations performed by Algorithm 4.1 from the root r of the tree to c inclusively. The following holds:

Proposition 5.2. *Let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a very-WSTS and let $I_0 \in \text{Idl}(X)$. Let $c, d \in \mathcal{T}_{I_0}$. The following holds:*

(1) *If $c \overset{*}{\dashrightarrow}_{\mathcal{T}} d$ and $\text{ideal}(c) = \text{ideal}(d)$, then $\text{num-accel}(c) = \text{num-accel}(d)$.*
(2) *If $c \overset{*}{\dashrightarrow}_{\mathcal{A}} d$, then $\text{num-accel}(c) \leq \text{num-accel}(d)$.*

Proof.

(1) For the sake of contradiction, suppose that $\text{num-accel}(c) \neq \text{num-accel}(d)$. This means that at least one acceleration occurred between c (exclusively) and d (inclusively). Let d' be the first accelerated node, i.e. the first node d' for which there exists c' such that $\text{num-accel}(d') = \text{num-accel}(c') + 1$ and

$$c \overset{+}{\dashrightarrow}_{\mathcal{T}} c' \dashrightarrow_{\mathcal{T}} d' \overset{*}{\dashrightarrow}_{\mathcal{T}} d.$$

By Proposition 3.11, $\text{lvl}(\text{ideal}(c)) = \text{lvl}(\text{ideal}(c')) < \text{lvl}(\text{ideal}(d')) \leq \text{lvl}(\text{ideal}(d))$. This is a contradiction since $\text{ideal}(c) = \text{ideal}(d)$.

(2) Since c can reach d, there exist a path of length $n \geq 0$ from c to d in $\mathcal{A}_{I_0}$. Let $c_0, c_1, \ldots c_n$ be the nodes visited by this path, where $c_0 = c$ and $c_n = d$. We prove the claim by induction on n. If $n = 0$, then $c = d$ and the the claim trivially holds. Assume that $n > 0$ and that the claims holds for paths of length $n - 1$. By induction hypothesis, $\text{num-accel}(c_0) \leq \text{num-accel}(c_{n-1})$. If c_n is an ancestor of c_{n-1} in $\mathcal{T}_{I_0}$ and $\text{ideal}(c_{n-1}) = \text{ideal}(c_n)$, then we are done since $\text{num-accel}(c_{n-1}) = \text{num-accel}(c_n)$ by (1). Otherwise, $c_{n-1} \overset{a}{\dashrightarrow}_{\mathcal{T}} c_n$ for some $a \in \Sigma$. By Proposition 3.11, $\text{num-accel}(c_{n-1}) \leq \text{num-accel}(c_n)$, and hence $\text{num-accel}(c_0) \leq \text{num-accel}(c_{n-1}) \leq \text{num-accel}(c_n)$. □

We may now prove the decidability of repeated coverability under some effectiveness hypotheses:

Theorem 5.3. *Repeated coverability is decidable for completion-post-effective, ∞-completion-effective and ideal-increasing-effective classes of very-WSTS.*

Proof. By Proposition 5.1, y is repeatedly coverable from x if and only if there exist $c \in \mathcal{A}_{\downarrow x}$ and $w \in \Sigma^+$ such that

$$c \xrightarrow{w}_{\mathcal{A}} c \text{ and } w \text{ is } (\mathrm{ideal}(c), y)\text{-increasing.} \tag{5.3}$$

We show how (5.3) can be tested. For every $c \in \mathcal{A}_{\downarrow x}$, let A_c be the automaton obtained from $\mathcal{A}_{\downarrow x}$ by taking c as the initial state and the unique accepting state. Let A_c^+ be a finite automaton that recognizes $L(A_c) \setminus \{\varepsilon\}$. By (5.3), y is repeatedly coverable from x if and only if there exists $c \in \mathcal{A}_{\downarrow x}$ such that

$$L(A_c^+) \text{ contains an } (\mathrm{ideal}(c), y)\text{-increasing sequence.} \tag{5.4}$$

Let us explain how to decide (5.4). First, note that A_c^+ can be constructed effectively for every c using the fact that $\mathcal{C}$ is completion-post-effective and ∞-completion-effective. We may also only consider nodes c such that $y \in \mathrm{ideal}(c)$. Note that if $L(A_c^+)$ contains an $(\mathrm{ideal}(c), y)$-increasing sequence, then $y \in \mathrm{Idl}(c)$ due to downward closure of ideals. Thus, when consider a node c, we may first test whether $y \in \mathrm{ideal}(c)$ by completion-post-effectiveness.

Now, to test (5.4), we may use the fact that $\mathcal{C}$ is ideal-increasing-effective. In order to do so, we must show that for every node c such that $y \in \mathrm{ideal}(c)$, the automaton A_c^+ is such that $\mathrm{Post}_{\widehat{\mathcal{S}}}(\mathrm{ideal}(c), w) \neq \emptyset$ for every $w \in L(A_c^+)$. Let c be such that $y \in \mathrm{ideal}(c)$ and let $w \in L(A_c^+)$. We have

$$c \xrightarrow{w}_{\mathcal{A}} c$$

and, by Proposition 5.2(2), no acceleration can occur along this path. Therefore, $\mathrm{ideal}(c) \overset{w}{\rightsquigarrow} \mathrm{ideal}(c)$ which implies that $\mathrm{Post}_{\widehat{\mathcal{S}}}(\mathrm{ideal}(c), w) \neq \emptyset$. □

Let us remark that ideal-increasing-effective holds for Petri nets and ω-Petri nets, since, for these models, testing whether a finite automaton A accepts some (I, x)-increasing sequence amounts to computing the Parikh image of $L(A)$, which is effectively semilinear [Par66]:

Proposition 5.4. *Petri nets (or vector addition systems with/without states) and ω-Petri nets are ideal-increasing-effective.*

Proof. Since ω-Petri nets encompass all three models, we only give a proof for that model. For a definition of ω-Petri nets, see either Section 4 or [GHPR15]. Let $\mathcal{S}$ be an ω-Petri net with places P and transitions T. Let $I \in \mathrm{Idl}(\mathbb{N}^P)$ and $\boldsymbol{x} \in I$. Let A be a finite automaton such that $\mathrm{Post}_{\widehat{\mathcal{S}}}(I, w) \neq \emptyset$ for every $w \in L(A)$. We will show how to determine whether $L(A)$ contains an $(I, \boldsymbol{x})$-increasing sequence. Before doing so, we introduce a few definitions.

For every place p and transition t of $\mathcal{S}$, let $\mathrm{Pre}(p,t)$ and $\mathrm{Post}(p,t)$ denote respectively the number of tokens consumed and produced by t in p. Let $\boldsymbol{N}$ be the matrix defined as follows:

$$\boldsymbol{N}(p,t) \stackrel{\text{def}}{=} \begin{cases} \mathrm{Post}(p,t) - \mathrm{Pre}(p,t) & \text{if } \mathrm{Post}(p,t) \neq \omega \text{ and } \mathrm{Pre}(p,t) \neq \omega, \\ \mathrm{Post}(p,t) & \text{otherwise.} \end{cases}$$

Intuitively, $\boldsymbol{N}$ records the maximal increment that can be achieved in each place by firing transitions of $\mathcal{S}$. In particular, if $\mathcal{S}$ is a standard Petri net, then $\boldsymbol{N}$ is its incidence matrix. By abuse of notation, "$z = y + \omega$" with $z, y \in \mathbb{N}$ will stand for "$z \geq y$".

For every word w, let Ψ_w be the *Parikh image* of w, i.e. the vector such that $\Psi_w(a)$ is the number of occurrences of a in w. Furthermore, let

$$\Psi_A \stackrel{\text{def}}{=} \{\Psi_w : w \in L(A)\},$$

and let $\boldsymbol{i} \stackrel{\text{def}}{=} \omega\text{-rep}(I)$, i.e. the vector from $\mathbb{N}_\omega^P$ associated to I.

We claim that there exists $w \in L(A)$ such that w is $(I, \boldsymbol{x})$-increasing if and only if there exist $\boldsymbol{p} \in \Psi_A$ and $\boldsymbol{y} \in \mathbb{N}^P$ such that

$$\boldsymbol{N} \cdot \boldsymbol{p} \geq \boldsymbol{0} \text{ and } \boldsymbol{x} \leq \boldsymbol{y} \leq \boldsymbol{i}. \tag{5.5}$$

Before proving the claim, let us see how it helps proving the proposition. By [Par66], it is possible to compute from A a Presburger-definable formula φ_A such that $\varphi_A(\boldsymbol{p})$ holds if and only if $\boldsymbol{p} \in \Psi_A$. Let $\varphi'(A, \boldsymbol{p}, \boldsymbol{i})$, written more simply φ', be the following Presburger-definable sentence[5]:

$$\exists \boldsymbol{p} \in \mathbb{N}^T, \exists \boldsymbol{y} \in \mathbb{N}^P : \varphi_A(\boldsymbol{p}) \wedge \boldsymbol{N} \cdot \boldsymbol{p} \geq \boldsymbol{0} \wedge \boldsymbol{x} \leq \boldsymbol{y} \leq \boldsymbol{i}.$$

By our claim, φ' holds if and only if $L(A)$ contains an $(I, \boldsymbol{x})$-increasing sequence. Thus, we derive an algorithm from the fact that φ' is effective and by decidability of Presburger arithmetic [Pre29] (see [BM07], e.g., for a modern presentation in English).

Let us now prove the claim.

($\Leftarrow$) Suppose there exist $\boldsymbol{p} \in \Psi_A$ and $\boldsymbol{y} \in \mathbb{N}^P$ such that (5.5) holds. Let $w \in L(A)$ be a word such that $\boldsymbol{p} = \Psi_w$. By hypothesis on A, w is fireable from I in $\widehat{S}$, i.e. $I \stackrel{w}{\rightsquigarrow} J$ for some J. Let $\boldsymbol{z} \in J$. By Proposition 3.4(2), there exist $\boldsymbol{y}' \in I$ and $\boldsymbol{z}' \geq \boldsymbol{z}$ such that $\boldsymbol{y}' \stackrel{w}{\to} \boldsymbol{z}'$. By (5.5), we have $\boldsymbol{y} \leq \boldsymbol{i}$. In other words, $\boldsymbol{y} \in I$. Since $\boldsymbol{y}'$ also belongs to I, which is a directed set, there exists $\boldsymbol{y}'' \in I$ such that $\boldsymbol{y}'' \geq \boldsymbol{y}$ and $\boldsymbol{y}'' \geq \boldsymbol{y}'$. By strong monotonicity of $\mathcal{S}$, we have $\boldsymbol{y}'' \stackrel{w}{\to} \boldsymbol{z}''$ for some $\boldsymbol{z}'' \geq \boldsymbol{z}'$. We claim that there exists $\boldsymbol{z}''' \geq \boldsymbol{z}''$ such that

$$\boldsymbol{z}''' = \boldsymbol{y}'' + \boldsymbol{N} \cdot \boldsymbol{p}. \tag{5.6}$$

Vector $\boldsymbol{z}'''$ can be derived by resolving the non determinism of each transition t occurring in the firing sequence w as follows:

- for every place p such that $\text{Pre}(p, t) = \omega$, we make every occurrence of t consume 0 token from p;
- for every place p such that $\text{Post}(p, t) = \omega$, we make every occurrence of t produce a sufficiently large amount of tokens in p, e.g. $|\boldsymbol{z}''(p) - \boldsymbol{y}''(p)|$ tokens.

This way, we have:

$$\begin{aligned} \boldsymbol{z}''' &= \boldsymbol{y}'' + \boldsymbol{N} \cdot \boldsymbol{p} && \text{(by (5.6))} \\ &\geq \boldsymbol{y}'' && \text{(by } \boldsymbol{N} \cdot \boldsymbol{p} \geq \boldsymbol{0} \text{ from (5.5)).} \end{aligned}$$

Moreover, by (5.5) and by transitivity, we have $\boldsymbol{x} \leq \boldsymbol{y} \leq \boldsymbol{y}''$. Thus, overall, we obtain $\boldsymbol{y}'' \stackrel{w}{\to} \boldsymbol{z}'''$ and $\boldsymbol{x} \leq \boldsymbol{y}'' \leq \boldsymbol{z}'''$, which means that w is $(I, \boldsymbol{x})$-increasing.

($\Rightarrow$) Suppose there exists $w \in L(A)$ such that w is $(I, \boldsymbol{x})$-increasing. By definition, there exist $\boldsymbol{y} \in I$ and $\boldsymbol{z} \in X$ such that $\boldsymbol{y} \stackrel{w}{\to} \boldsymbol{z}$ and $\boldsymbol{x} \leq \boldsymbol{y} \leq \boldsymbol{z}$. Let us take $\boldsymbol{p} \stackrel{\text{def}}{=} \Psi_w$. By definition of $\boldsymbol{N}$, the following holds for ω-Petri nets (and it is an equality for Petri nets):

$$\boldsymbol{z} \leq \boldsymbol{y} + \boldsymbol{N} \cdot \Psi_w. \tag{5.7}$$

[5]Note that Presburger arithmetic typically only allows for integers coefficients, while $\boldsymbol{N}$ and $\boldsymbol{i}$ may contain ω's. However, this is not an issue since constraints of the form "$\sum_{i, a_i \in \mathbb{Z}} a_i \cdot x_i + \sum_j \omega \cdot x_j \geq 0$" and "$x \leq \omega$" can respectively be replaced by "$\sum_{i, a_i \in \mathbb{Z}} a_i \cdot x_i \geq 0 \vee \bigvee_j x_j \geq 0$" and "*true*".

Thus, by (5.7), we have $\boldsymbol{y} \leq \boldsymbol{z} \leq \boldsymbol{y} + \boldsymbol{N} \cdot \boldsymbol{p}$. This implies that $\boldsymbol{N} \cdot \boldsymbol{p} \geq \boldsymbol{0}$. Moreover, the inequalities $\boldsymbol{x} \leq \boldsymbol{y} \leq \boldsymbol{i}$ hold by hypothesis and by the fact that $\boldsymbol{y} \in I$ which is equivalent to $\boldsymbol{y} \leq \boldsymbol{i}$. □

5.2. **From model checking to repeated coverability.** We conclude this section by reducing LTL model checking to repeated coverability. Recall that a *Büchi automaton* $\mathcal{B}$ is a non-deterministic finite automaton $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ interpreted over Σ^ω. An infinite word is accepted by $\mathcal{B}$ if it contains an infinite path from q_0 labeled by w and visiting F infinitely often. We denote by $\mathrm{L}(\mathcal{B})$ the set of infinite words accepted by $\mathcal{B}$.

Let $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ be a Büchi automaton and let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a WSTS. The product of $\mathcal{B}$ and $\mathcal{S}$ is defined as

$$\mathcal{B} \times \mathcal{S} \stackrel{\text{def}}{=} (Q \times X, \xrightarrow{\Sigma \times Q}, = \times \leq)$$

where $(p, x) \xrightarrow{(a,r)} (q, y)$ if $(p, a, r) \in \delta, q = r$ and $x \xrightarrow{a} y$. The point in including r in the label is so that the completion of $\mathcal{B} \times \mathcal{S}$ is deterministic, a requirement for very-WSTS. This is formalized in the following proposition:

Proposition 5.5. *Let $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ be a Büchi automaton and let $\mathcal{S} = (X, \xrightarrow{\Sigma}, \leq)$ be a very-WSTS. The product $\mathcal{B} \times \mathcal{S}$ is a very-WSTS. Moreover, it preserves completion-post-effectiveness, ∞-completion-effectiveness and ideal-increasing-effectiveness.*

Proof. Let us show that $\mathcal{B} \times \mathcal{S}$ is a WSTS with strong monotonicity. Since equality is a wqo for finite sets and since wqos are closed under cartesian product, $= \times \leq$ is a wqo. Let $p, q \in Q$, $x, x', y \in X$ and $(a, r) \in \Sigma \times Q$ be such that

$$(p, x) \xrightarrow{(a,r)} (q, y) \text{ and } x' \geq y.$$

By definition of $\mathcal{B} \times \mathcal{S}$, we have $(p, a, q) \in \delta$, $r = q$ and $x \xrightarrow{a} y$. By strong monotonicity of $\mathcal{S}$, there exists $y' \geq y$ such that $x' \xrightarrow{a} y'$. Therefore,

$$(p, x') \xrightarrow{(a,r)} (q, y').$$

It remains to show that the completion of $\mathcal{B} \times \mathcal{S}$ is a deterministic WSTS with leveled-strong-strict monotonicity, and that $\mathrm{Idl}(Q \times X)$ has finitely many levels. First note that

$$\mathrm{Idl}(Q \times X) = \{\{q\} \times I : q \in Q, I \in \mathrm{Idl}(X)\}. \tag{5.8}$$

Since $\mathrm{Idl}(X)$ has finitely many levels, it follows from (5.8) that $\mathrm{Idl}(Q \times X)$ also has finitely many levels. Similarly, $\mathrm{Idl}(Q \times X)$ is well-quasi-ordered by $\subseteq$ since $\mathrm{Idl}(X)$ is well-quasi-ordered by $\subseteq$ and since Q is finite. We also note that ideal levels are preserved, i.e. $\mathrm{lvl}(\{q\} \times I) = \mathrm{lvl}(I)$ for every $q \in Q$ and $I \in \mathrm{Idl}(X)$.

Leveled-strong-strict monotonicity. Let $I, I', J \in \mathrm{Idl}(Q \times X)$, $a \in \Sigma$ and $r \in Q$ be such that $\mathrm{lvl}(I) \neq \infty$ and

$$I \subset I', I \overset{(a,r)}{\rightsquigarrow} J \text{ and } \mathrm{lvl}(I) = \mathrm{lvl}(J).$$

By (5.8), there exist $p, q \in Q$ and $I_p, I'_p, J_q \in \mathrm{Idl}(X)$ such that $I = \{p\} \times I_p$, $I' = \{p\} \times I'_p$ and $J = \{q\} \times J_q$. We have $I_p \subset I'_p$, $I_p \overset{a}{\rightsquigarrow} J_q$, $q = r$ and $(p, a, r) \in \delta$. By leveled-strong-strict monotonicity of $\widehat{\mathcal{S}}$, there exists $J'_q \in \mathrm{Idl}(X)$ such that $I'_p \overset{a}{\rightsquigarrow} J'_q$ and $J_q \subset J'_q$. Let $J' \overset{\text{def}}{=} \{q\} \times J'_q$. We obtain

$$I' \overset{(a,r)}{\rightsquigarrow} J' \text{ and } J \subset J'.$$

Determinism. Let $I, J, J' \in \mathrm{Idl}(Q \times X)$, $a \in \Sigma$ and $r \in Q$ be such that $I \overset{(a,r)}{\rightsquigarrow} J$ and $I \overset{(a,r)}{\rightsquigarrow} J'$. By (5.8), there exist $p, q, q' \in Q$ and $I_p, J_q, J_{q'} \in \mathrm{Idl}(X)$ such that $I = \{p\} \times I_p$, $J = \{q\} \times J_q$ and $J' = \{q'\} \times J_{q'}$. We have $r = q = q'$, $I_p \overset{a}{\rightsquigarrow} J_q$ and $I_p \overset{a}{\rightsquigarrow} J_{q'}$. Since $\widehat{\mathcal{S}}$ is deterministic, we have $J_q = J_{q'}$, and hence $J = J'$.

Effectivenesses. Completion-post-effectiveness is preserved due to the fact that: ideals can be represented by an extra finite state; testing $\{p\} \times I \subseteq \{q\} \times J$ simply amounts to testing whether $p = q$ and $I \subseteq J$; and computing $\mathrm{Post}_{\widehat{\mathcal{S}}}(\{q\} \times I, (a, r))$ simply amounts to computing the successors of q and I under a in $\mathcal{B}$ and $\mathcal{S}$ respectively. The ∞-completion-effectiveness is preserved due to the fact that any acceleration candidate must, by definition, be of the form $\{q\} \times I_0, \{q\} \times I_1, \{q\} \times I_2, \ldots$ for some $q \in Q$, and hence that it suffices to perform accelerations in $\widehat{S}$. Similarly, ideal-increasing-effectiveness is preserved because testing whether a sequence w is $(\{p\} \times I, (q, x))$-increasing amounts to testing whether w is (I, x)-increasing and whether $p = q$; this follows again from the fact that Q is ordered by equality. □

For every WSTS $\mathcal{S} = (X, \overset{\Sigma}{\rightarrow}, \leq)$, we extend $\mathcal{S}$ with a new "minimal" element $\perp$ smaller than every other states, i.e.

$$\mathcal{S}_\perp \overset{\text{def}}{=} (X \cup \{\perp\}, \overset{\Sigma}{\rightarrow}, \leq_\perp)$$

where transition relations are unchanged, and $\leq_\perp \overset{\text{def}}{=} \leq \cup \{(\perp, y) : y \in X \cup \{\perp\}\}$. Adding the minimal element $\perp$ preserves the properties of very-WSTS:

Proposition 5.6. *$\mathcal{S}_\perp$ is a very-WSTS for every very-WSTS $\mathcal{S}$. Moreover, it preserves completion-post-effectiveness, ∞-completion-effectiveness and ideal-increasing-effectiveness.*

Proof. It is readily seen that $X \cup \{\perp\}$ is a wqo and that $\mathcal{S}_\perp$ preserves the strong monotonicity of $\mathcal{S}$. Note that $\mathrm{Idl}(X \cup \{\perp\}) = \{\perp\} \cup \{I \cup \{\perp\} : I \in \mathrm{Idl}(X)\}$. Since inclusion is a wqo for $\mathrm{Idl}(X)$, it is also a wqo for $\mathrm{Idl}(X \cup \{\perp\})$. Morever, $\mathrm{Idl}(X \cup \{\perp\})$ has as many levels as $\mathrm{Idl}(X)$. Let $\rightsquigarrow_\perp$ denote the transition relation of the completion of $\mathcal{S}_\perp$. For every $I, J \in \mathrm{Idl}(X)$ and $a \in \Sigma$, we have $I \overset{a}{\rightsquigarrow} J$ if and only if $I \cup \{\perp\} \overset{a}{\rightsquigarrow}_\perp J \cup \{\perp\}$. Therefore, the completion of $\mathcal{S}_\perp$ is also deterministic and also has leveled-strong-strict monotonicity. It is readily seen that effectivenesses are preserved. □

Taking the product of $\mathcal{B}$ and $\mathcal{S}_\perp$ allows us to test whether a word of $\mathrm{L}(\mathcal{B})$ is also an infinite trace of $\mathcal{S}$:

Proposition 5.7. *Let $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ be a Büchi automaton, let $\mathcal{S} = (X, \overset{\Sigma}{\rightarrow}, \leq)$ be a very-WSTS, and let $x_0 \in X$. There exists $w \in \mathrm{L}(\mathcal{B}) \cap \omega\text{-Traces}_{\mathcal{S}}(x_0)$ if and only if there exists $q_f \in F$ such that $(q_f, \perp)$ is repeatedly coverable from (q_0, x_0) in $\mathcal{B} \times \mathcal{S}_\perp$.*

Proof. ($\Rightarrow$) Let $w \in \mathrm{L}(\mathcal{B}) \cap \omega\text{-Traces}_{\mathcal{S}}(x_0)$. Since $w \in \mathrm{L}(\mathcal{B})$, there exist $q_1, q_2, \ldots \in Q$ such that $q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} q_2 \xrightarrow{w_3} \cdots$ and $q_i \in F$ for infinitely many $i \in \mathbb{N}$. Since F is finite, there exists some $q_f \in F$ such that $q_i = q_f$ for infinitely many $i \in \mathbb{N}$. Since $w \in \omega\text{-Traces}_{\mathcal{S}}(x_0)$, there exist $x_1, x_2, \ldots \in X$ such that $x_0 \xrightarrow{w_1} x_1 \xrightarrow{w_2} x_2 \xrightarrow{w_3} \cdots$. Therefore,

$$(q_0, x_0) \xrightarrow{(w_1,q_1)} (q_1, x_1) \xrightarrow{(w_2,q_2)} (q_2, x_2) \xrightarrow{(q_3,w_3)} \cdots$$

which implies that $(q_f, \bot)$ is repeatedly coverable from (q_0, x_0) in $\mathcal{B} \times \mathcal{S}_\bot$ since $x_i \geq \bot$ for every $i \in \mathbb{N}$.

($\Leftarrow$) Suppose $(q_f, \bot)$ is repeatedly coverable from (q_0, x_0) in $\mathcal{B} \times \mathcal{S}_\bot$. There exist $(a_1, q_1), (a_2, q_2), \ldots \in \Sigma \times Q$ and $(q_1, x_1), (q_2, x_2), \ldots \in Q \times X$ such that

$$(q_0, x_0) \xrightarrow{(a_1,q_1)} (q_1, x_1) \xrightarrow{(a_2,q_2)} (q_2, x_2) \xrightarrow{(a_3,q_3)} \cdots \tag{5.9}$$

and $q_i = q_f$ and $x_i \geq \bot$ for infinitely many $i \in \mathbb{N}$. By (5.9) and by definition of $\mathcal{B} \times \mathcal{S}_\bot$, we have $(q_i, a_i, q_{i+1}) \in \delta$ and $x_i \xrightarrow{a_i} x_{i+1}$ for every $i \in \mathbb{N}$. Therefore, we conclude that $a_1 a_2 \cdots \in \mathrm{L}(\mathcal{B}) \cap \omega\text{-Traces}_{\mathcal{S}}(x_0)$. □

Theorem 5.3 together with Propositions 5.5, 5.6 and 5.7 imply the decidability of LTL model checking:

Theorem 5.8. *LTL model checking is decidable for completion-post-effective, ∞-completion-effective and ideal-increasing-effective classes of very-WSTS.*

Theorem 5.8 implies that LTL model checking for ω-Petri nets is decidable. This includes and strictly generalizes decidability of termination in ω-Petri nets [GHPR15] and decidability of LTL model checking for Petri nets.

To the best of our knowledge, we also provide the first self-contained presentation of the decidability of LTL model checking for Petri nets that does not rely on Rackoff techniques. The first proof for the decidability of LTL model checking for Petri nets comes from Esparza [Esp94]; it uses a result from Jantzen and Valk [VJ85] on the decidability of the existence of an infinite number of occurrences of a given transition in an infinite run. This essentially corresponds to our general study of (I, x)-increasing sequences. Moreover, to derive a 2-EXPSPACE complexity bound, Esparza also used the logic of Yen [Yen92] which extends Rackoff techniques. Unfortunately, some flaws in the paper of Yen were found later by Atig and Habermehl [AH11]. Habermehl gave the first proof that the linear-time μ-calculus is in EXPSPACE [Hab97]; his proof directly uses techniques *a la Rackoff* to compute the length of short witnesses of some infinite runs. In both previous papers, on the decidability of LTL model checking for Petri nets, it is not clear how the proofs found therein can be extended to general very-WSTS.

6. A characterization of acceleration levels

In this section, we give a precise characterization of ideals that have finitely many levels.

Let us redefine the family of sets $A_n(\mathrm{Idl}(X))$ introduced in Section 3.2 in a more general setting. Let Z be a well-founded partially ordered set, abstracting away from the case $Z = \mathrm{Idl}(X)$. We say that a sequence $z_0, z_1, \ldots \in Z$ is an *acceleration candidate* if $z_1 < z_2 < \cdots$. Such an acceleration candidate is *below* $z \in Z$ if $z_i \leq z$ for every $i \in \mathbb{N}$, and *goes through* a set A if $z_i \in A$ for some $i \in \mathbb{N}$.

Definition 6.1. Let Z be a partially ordered set. Let $A_0(Z) \stackrel{\text{def}}{=} \emptyset$. For every ordinal $\alpha > 0$, $A_\alpha(Z)$ is the set of elements $z \in Z$ such that every acceleration candidate below z goes through $A_\beta(Z)$ for some $\beta < \alpha$.

The observations made in Section 3.2 still hold, i.e. $A_\alpha(Z) \subseteq A_\beta(Z)$ and $A_\alpha(Z)$ is downward-closed for every $\alpha \leq \beta$.

The *rank* of $z \in Z$, denoted $\operatorname{rk} z$, is the ordinal defined inductively by

$$\operatorname{rk} z \stackrel{\text{def}}{=} \sup\{\operatorname{rk} y + 1 : y < z\},$$

where $\sup(\emptyset) \stackrel{\text{def}}{=} 0$. The *rank* of Z is defined as

$$\operatorname{rk} Z \stackrel{\text{def}}{=} \sup\{\operatorname{rk} z + 1 : z \in Z\}.$$

Let us first show that $A_n(Z)$ is exactly the set of elements of rank less than $\omega \cdot n$. This rests on the following, which is perhaps less obvious than it seems.

Lemma 6.2. *Let Z be a countable wpo. For every $z \in Z$ such that $\operatorname{rk} z$ is a limit ordinal, z is the supremum of some acceleration candidate $z_0 < z_1 < \cdots$. Moreover, for any given ordinal $\beta < \operatorname{rk} z$, the acceleration candidate can be chosen such that $\beta \leq z_i$ for every $i \in \mathbb{N}$.*

Proof. Let $\alpha \stackrel{\text{def}}{=} \operatorname{rk} z$. A *fundamental sequence* for α is a monotone sequence of ordinals strictly below α whose supremum equals α. Fundamental sequences exist for all countable limit ordinals, in particular for α, since Z is countable (e.g. see [For10]). Pick one such fundamental subsequence $(\gamma_i)_{i\in\mathbb{N}}$. Replacing γ_i by $\sup(\beta, \gamma_i)$ if necessary, we may assume that $\beta \leq \gamma_m$ for every $i \in \mathbb{N}$. By the definition of rank, for every $i \in \mathbb{N}$, there is an element $z_i < z$ of rank at least γ_i. Since Z is well-quasi-ordered, we may extract a non-decreasing subsequence from $(z_i)_{i\in\mathbb{N}}$. Without loss of generality, assume that $z_0 \leq z_1 \leq \cdots$. If all but finitely many of these inequalities were equalities, then z would be equal to z_i for m large enough, but that is impossible since $z_i < z$. We can therefore extract a strictly increasing subsequence from $(z_i)_{i\in\mathbb{N}}$. This is an acceleration sequence, its supremum is z, and $\beta \leq \gamma_i \leq z_i$ for every i. □

Note that Lemma 6.2 fails if Z is not countable: take $Z = \omega_1 + 1$, where ω_1 is the first uncountable ordinal, then $\omega_1 \in Z$ is not the supremum of countably many ordinals $< \omega_1$. This also fails if Z is not well-quasi-ordered, even when Z is well-founded: consider the set with one root r above chains of length n, one for each $n \in \mathbb{N}$: $\operatorname{rk} r = \omega$, but there is no acceleration candidate below r.

Lemma 6.3. *Let Z be a countable wpo, and let $n \in \mathbb{N}$. For every $z \in Z$, $\operatorname{rk} z < \omega \cdot n$ if and only if $z \in A_n(Z)$.*

Proof. ($\Rightarrow$) By induction on n. The case $n = 0$ is immediate. Let $n \geq 1$. Given any acceleration candidate $z_1 < z_2 < \cdots$ below z, we must have $\operatorname{rk} z_1 < \operatorname{rk} z_2 < \cdots < \operatorname{rk} z$. Since $\operatorname{rk} z < \omega \cdot n$, there exist $\ell, m \in \mathbb{N}$ with $\ell < n$ such that $\operatorname{rk} z = \omega.\ell + m$. Therefore, $\operatorname{rk} z_i \geq \omega \cdot \ell$ for only finitely many i. In particular, there exists some i such that $\operatorname{rk} z_i < \omega \cdot \ell$. Since $\ell < n$, we have $\operatorname{rk} z_i < \omega \cdot (n-1)$. By induction hypothesis, $z_i \in A_{n-1}(Z)$, and hence $z \in A_n(Z)$.

($\Leftarrow$) We show by induction on n that $\operatorname{rk} z \geq \omega \cdot n$ implies $z \notin A_n(Z)$. The case $n = 0$ is immediate. Let $n \geq 1$. In general, $\operatorname{rk} z$ is not a limit ordinal, but can be written as $\alpha + \ell$ for some limit ordinal α and some $\ell \in \mathbb{N}$. By definition of rank, z is larger than some element of rank $\alpha + (\ell - 1)$, which is itself larger than some element of rank $\alpha + (\ell - 2)$, and so on. Iterating this way, we find an element $y \leq z$ of rank exactly α. Since $\operatorname{rk} y$ is a limit ordinal,

Lemma 6.2 entails that y is the supremum of some acceleration candidate $z_0 < z_1 < \cdots$. Moreover, since $\omega \cdot (n-1) < \operatorname{rk} y$, we may assume that $\operatorname{rk} z_i \geq \omega \cdot (n-1)$ for every $i \in \mathbb{N}$. By induction hypothesis, $z_i \notin A_{n-1}(Z)$ for every $i \in \mathbb{N}$, and hence $z \notin A_n(Z)$. □

Theorem 6.4. *Let X be a countable wqo such that $\operatorname{Idl}(X)$ is well-quasi-ordered by inclusion*[6]*. The following holds: $\operatorname{Idl}(X)$ has finitely many levels if and only if $\operatorname{rk}\operatorname{Idl}(X) < \omega^2$.*

Proof. We apply Lemma 6.3 to $Z = \operatorname{Idl}(X)$, a wpo by assumption. For that, we need to show that Z is countable. There are countably many upward-closed subsets, since they are all determined by their finitely many minimal elements. Downward-closed subsets are in one-to-one correspondence with upward-closed subsets, through complementation, hence are countably many as well, and ideals are particular downward-closed subsets.

We conclude by noting that the following are equivalent: (1) $\operatorname{rk}\operatorname{Idl}(X) < \omega^2$; (2) $\operatorname{rk}\operatorname{Idl}(X) \leq \omega \cdot n$ for some $n \in \mathbb{N}$; (3) $A_n(\operatorname{Idl}(X)) = \operatorname{Idl}(X)$ for some $n \in \mathbb{N}$ (by Lemma 6.3); (4) $A_n(X) = \emptyset$ for some $n \in \mathbb{N}$. □

While $\operatorname{rk}\operatorname{Idl}(\mathbb{N}^d) = \omega \cdot d + 1 < \omega^2$, not all wqos X used in formal verification satisfy $\operatorname{rk}\operatorname{Idl}(X) < \omega^2$. For example, $\operatorname{rk}\operatorname{Idl}(\Sigma^*) = \omega^{|\Sigma|} + 1$, for any finite alphabet Σ; a similar result holds for multisets over Σ.

Proposition 6.5. $\operatorname{rk}\operatorname{Idl}(\Sigma^*) = \omega^{|\Sigma|} + 1$ *for every finite alphabet Σ.*

Proof. Let $k \stackrel{\text{def}}{=} |\Sigma|$. The elements of $\operatorname{Idl}(\Sigma^*)$ are *word-products* P, defined as formal products $e_1 e_2 \cdots e_m$ of atomic expressions of the form $a^?$, $a \in \Sigma$, or A^*, where $a^?$ denotes $\{a, \varepsilon\}$ and A is a non-empty subset of Σ [KP92, FG09]. Word-products were introduced under this name in [ACABJ04].

Lower bound. Enumerate the letters of Σ as a_1, a_2, …, a_k. Let $A_i = \{a_1, a_2, \ldots, a_i\}$. Any ordinal α strictly less than ω^k can be written in a unique way as $\omega^{k-1} \cdot n_{k-1} + \omega^{k-2} \cdot n_{k-2} + \cdots + \omega \cdot n_1 + n_0$. Define an ideal I_α by the word-product

$$(a_1^?)^{n_0} (a_2^? A_1^*)^{n_1} (a_3^? A_2^*)^{n_2} \cdots (a_k^? A_{k-1}^*)^{n_{k-1}}.$$

The first terms, n_0 times $a_1^?$, have a different format from the rest of the word-product. For uniformity of treatment, we write $a_1^?$ as $a_1^? A_0^*$ (indeed $A_0^* = \emptyset^* = \{\epsilon\}$), so $I_\alpha = (a_1^? A_0^*)^{n_0} (a_2^? A_1^*)^{n_1} \cdots (a_k^? A_{k-1}^*)^{n_{k-1}}$.

We claim that $\beta > \alpha$ implies $I_\beta \supset I_\alpha$.

Let $\alpha = \omega^{k-1} \cdot n_{k-1} + \omega^{k-2} \cdot n_{k-2} + \cdots + \omega \cdot n_1 + n_0$ and $\beta = \omega^{k-1} \cdot m_{k-1} + \omega^{k-2} \cdot m_{k-2} + \cdots + \omega \cdot m_1 + m_0$. The condition $\beta > \alpha$ is equivalent to the fact that $(m_{k-1}, m_{k-2}, \ldots, m_1, m_0)$ is lexicographically larger than $(n_{k-1}, n_{k-2}, \ldots, n_1, n_0)$. Write $\beta \to \alpha$ if for some i with $0 \leq i < k$, $n_{k-1} = m_{k-1}$, $n_{k-2} = m_{k-2}$, …, $n_{i+1} = m_{i+1}$, and $m_i = n_i + 1$. Since $>$ is the transitive closure of $\to$, it suffices to show that $\beta \to \alpha$ implies $I_\beta \supset I_\alpha$.

Containment is proved as follows. $I_\beta = (a_1^? A_0^*)^{m_0} (a_2^? A_1^*)^{m_1} \cdots (a_k^? A_{k-1}^*)^{m_{k-1}}$ contains $(a_{i+1}^? A_i^*)^{m_i} (a_{i+2}^? A_{i+1}^*)^{m_{i+1}} \cdots (a_k^? A_{k-1}^*)^{m_{k-1}}$, because the empty word belongs to the removed prefix $(a_1^? A_0^*)^{m_0} (a_2^? A_1^*)^{m_1} \cdots (a_i^? A_{i-1}^*)^{m_{i-1}}$. Since $m_i = n_i + 1 \geq 1$, we can write $(a_{i+1}^? A_i^*)^{m_i} (a_{i+2}^? A_{i+1}^*)^{m_{i+1}} \cdots (a_k^? A_{k-1}^*)^{m_{k-1}}$ as $a_{i+1}^? A_i^* P$, where P abbreviates $(a_{i+1}^? A_i^*)^{n_i} (a_{i+2}^? A_{i+1}^*)^{m_{i+1}} \cdots (a_k^? A_{k-1}^*)^{m_{k-1}}$. Hence I_β contains $A_i^* P$. By the definition of $\to$, P is equal to

$$(a_{i+1}^? A_i^*)^{n_i} (a_{i+2}^? A_{i+1}^*)^{n_{i+1}} \cdots (a_k^? A_{k-1}^*)^{n_{k-1}}.$$

[6] Recall that such a wqo is known as an ω^2-wqo [FG12]. That we find the ordinal ω^2 in the statement of Theorem 6.4 and in the notion of ω^2-wqo seems to be coincidental.

We now note that A_i^* contains $(a_1^? A_0^*)^{n_0}(a_2^? A_1^*)^{n_1}(a_3^? A_2^*)^{n_2}\cdots(a_i^? A_{i-1}^*)^{n_{i-1}}$, because every word in the latter contains only letters from $\{a_1, a_2, \ldots, a_i\} = A_i$. Hence $A_i^* P$ contains $(a_1^? A_0^*)^{n_0}(a_2^? A_1^*)^{n_1}(a_3^? A_2^*)^{n_2}\cdots(a_i^? A_{i-1}^*)^{n_{i-1}} P$, which is equal to I_α. Since I_β contains $A_i^* P$, we conclude.

We now show that containment is strict. Let w be the word $a_1^{m_0}(a_2a_1)^{m_1}(a_3a_2)^{m_2}\cdots(a_ka_{k-1})^{m_{k-1}}$. Clearly, w is in I_β. To show that w is not in I_α, we show that $u(a_{i+1}a_i)^{n_i+1}(a_{i+2}a_{i+1})^{m_{i+1}}\cdots(a_ja_{j-1})^{m_{j-1}}$ is not in $L(a_{i+1}^? A_i^*)^{n_i}(a_{i+2}^? A_{i+1}^*)^{m_{i+1}}\cdots(a_j^? A_{j-1}^*)^{m_{j-1}}$ for any j, $i+1 \leq j \leq k$, where u is an arbitrary word in A_i^* and L is an arbitrary language included in A_i^*. We will obtain $w \notin I_\alpha$ by letting $j = k$, $u = a_1^{m_0}(a_2a_1)^{m_1}\cdots(a_ia_{i-1})^{m_{i-1}}$ and $L = (a_1^? A_0^*)^{n_0}(a_2^? A_1^*)^{n_1}(a_3^? A_2^*)^{n_2}\cdots(a_i^? A_{i-1}^*)^{n_{i-1}}$. This is by induction on $j-(i+1)$. If $j = i+1$, we must show that $u(a_{i+1}a_i)^{n_i+1}$ is not in $L(a_{i+1}^? A_i^*)^{n_i}$, and that is obvious since any word in $L(a_{i+1}^? A_i^*)^{n_i}$ can contain at most n_i occurrences of a_{i+1}. In the induction case, let $v = u(a_{i+1}a_i)^{n_i+1}(a_{i+2}a_{i+1})^{m_{i+1}}\cdots(a_ja_{j-1})^{m_{j-1}}$, $A = L(a_{i+1}^? A_i^*)^{n_i}(a_{i+2}^? A_{i+1}^*)^{m_{i+1}}\cdots(a_j^? A_{j-1}^*)^{m_{j-1}}$, and let us show that $v(a_{j+1}a_j)^{m_j} \notin A(a_{j+1}^? A_j^*)^{m_j}$, knowing that $v \notin A$ by induction hypothesis. If $v(a_{j+1}a_j)^{m_j}$ were in $A(a_{j+1}^? A_j^*)^{m_j}$, there would be two words $v_1 \in A$ and $v_2 \in (a_{j+1}^? A_j^*)^{m_j}$ such that $v(a_{j+1}a_j)^{m_j} = v_1v_2$. Since v_2 is a suffix of $v(a_{j+1}a_j)^{m_j}$ and is in $(a_{j+1}^? A_j^*)^{m_j}$, v_2 must in fact be a suffix of $(a_{j+1}a_j)^{m_j}$. Hence v_1 contains v as prefix. However, v_1 is in A and A is downward-closed, and that implies $v \in A$ in particular: contradiction.

This ends our proof that $\beta > \alpha$ implies $I_\beta \supset I_\alpha$. Since $I_\beta \supset I_\alpha$ implies $\operatorname{rk} I_\beta > \operatorname{rk} I_\alpha$, an easy ordinal induction shows that $\operatorname{rk} I_\alpha \geq \alpha$ for every ordinal $\alpha < \omega^k$. There is a further ideal $A_k^* = \Sigma^*$ in Σ^*. It contains every I_α, and strictly so since the number of occurrences of a_k in any word of I_α is bounded from above by n_{k-1} (where $\alpha = \omega^{k-1}\cdot n_{k-1} + \omega^{k-2}\cdot n_{k-2} + \cdots + \omega\cdot n_1 + n_0$), but there are words with arbitrarily many occurrences of a_k in A_k^*. It follows that the rank of A_k^* in $\mathrm{Idl}(\Sigma^*)$ is at least $\sup\{\alpha+1 \mid \alpha < \omega^k\} = \omega^k$, and therefore that the rank of $\mathrm{Idl}(\Sigma^*)$ is at least $\omega^k + 1$.

Upper bound. Order atomic expressions by: $A^* \sqsubset B^*$ if and only if $A \subset B$, $a^? \sqsubset B^*$ if and only if $a \in B$, and no other strict inequality holds. The relation $\sqsubset$ is simply strict inclusion of the corresponding ideals. A word-product $P = e_1e_2\cdots e_m$ is *reduced* if and only if the ideal e_ie_{i+1} is included neither in e_i nor in e_{i+1}, for every i, $1 \leq i < m$. Reduced word-products are normal forms for word-products [ACABJ04]. On reduced word-products, we define two binary relations $\sqsubset^{\mathrm{w}}$ and $\sqsubseteq^{\mathrm{w}}$ by the following rules, and the specification that $\sqsubseteq^{\mathrm{w}}$ is the reflexive closure of $\sqsubset^{\mathrm{w}}$:

$$\frac{eP \sqsubseteq^{\mathrm{w}} P'}{eP \sqsubset^{\mathrm{w}} e'P'} \qquad \frac{P \sqsubset^{\mathrm{w}} P'}{a^?P \sqsubset^{\mathrm{w}} a^?P'} \qquad \frac{\forall i \cdot e_i \sqsubset A'^* \quad P \sqsubseteq^{\mathrm{w}} P'}{e_1\ldots e_kP \sqsubset^{\mathrm{w}} A'^*P'} \qquad \frac{P \sqsubset^{\mathrm{w}} P'}{A^*P \sqsubset^{\mathrm{w}} A^*P'}$$

Those rules are taken from [Gou13, Figure 1], and specialized to the case where all letters from Σ are incomparable. (That means that the rule called (w2) there never applies, and we have kept the remaining rules (w1), (w3)–(w5).) For reduced word-products P and P', $P \sqsubset^{\mathrm{w}} P'$ if and only if P, as an ideal, is strictly contained in P' (loc.cit.; alternatively, this is an easy exercise from the characterization of [non-strict] inclusion in [ACABJ04].) It follows that if P is strictly below P' in $\mathrm{Idl}(\Sigma^*)$, then $\mu(P)$ is strictly below $\mu(Q)$ in the multiset extension of $\sqsubset$, where, for $P = e_1e_2\cdots e_m$, $\mu(P)$ is the multiset $\{e_1, e_2, \ldots, e_m\}$, a fact already used in [Gou13].

The set of atomic expressions consists of the following elements: elements of the form $a^?$ are at the bottom, and have rank 1; just above, we find $\{a\}^*$, of rank 2, then $\{a,b\}^*$ of rank 3, etc.. In other words, A^* has rank one plus the cardinality of A. In particular, all atomic expressions except Σ^* have rank at most k.

Among reduced word-products P, those that are different from Σ^* must be of the form $e_1e_2\cdots e_m$ where no e_i is equal to Σ^*. This is by definition of reduction. Hence the suborder of those reduced word-products $P \neq \Sigma^*$ has rank less than or equal to the set of multisets $\{e_1, e_2, \ldots, e_m\}$ where each e_i has rank at most k (in the set of atomic expressions different from Σ^*).

The rank of the set of multisets of elements, where each element has rank at most k, is at most ω^k. This is well-known, but here is a short argument. We can map any multiset $\{e_1, e_2, \ldots, e_m\}$ to the ordinal $\omega^{k-1}\cdot n_{k-1} + \omega^{k-2}\cdot n_{k-2} + \cdots + \omega\cdot n_1 + n_0$ where n_i counts the number of elements e_j of rank i, and we observe that this mapping is strictly monotone.

It follows that the suborder of those reduced word-products P that are different from Σ^* has rank at most ω^k. $\mathrm{Idl}(\Sigma^*)$ contains just one additional element, Σ^*, which is therefore of rank at most ω^k. Hence $\mathrm{Idl}(\Sigma^*)$ has rank at most $\omega^k + 1$. □

7. Discussion and further work

We have presented the framework of very-WSTS, for which we have given a Karp-Miller algorithm. This allowed us to show that ideal decompositions of coverability sets of very-WSTS are computable, and that LTL model checking is decidable under some additional assumptions. We have also characterized acceleration levels in terms of ordinal ranks. Finally, we have shown that downward traces inclusion is decidable for very-WSTS.

As future work, we propose to study well-structured models beyond very-WSTS for which there exist Karp-Miller algorithms, e.g. unordered data Petri nets (UDPN) [HMM14, HLL$^+$16], or for which reachability is decidable, e.g. recursive Petri nets[7] [HP07] with strict monotonicity. It is conceivable that LTL model checking is decidable for such models. Our approach will have to be extended to tackle this problem.

For example, UDPN do not have finitely many acceleration levels. To circumvent this issue, Hofman et al. [HLL$^+$16] make use of two types of accelerations that can be nested. One type is prioritized to ensure that acceleration levels along a branch grow "fast enough" for the algorithm to terminate. A possible way to apply the theory of very-WSTS to such models that are not very-WSTS simply because they have an ideal rank larger or equal to ω^2 could be to find an abstraction of the set of ideals that reduces their rank to an ordinal strictly smaller that ω^2 while preserving suitable acceleration properties.

Moreover, observe that the IKM algorithm still terminates if, for each branch $B = (c_0\colon I_0, c_1\colon I_1, \ldots)$ of the Ideal Karp-Miller tree, the following set has rank less than ω^2:

$$[B] \stackrel{\text{def}}{=} \{I \in \mathrm{Idl}(X) : \exists j,k \in \mathbb{N}, j \leq k \text{ and } I_j \subseteq I \subseteq I_k\}.$$

Indeed, the IKM algorithm terminates if and only if each branch B is finite, and the states involved in computing the branch, as well as all needed accelerations, are all included in $[B]$. Therefore, relaxing "$\mathrm{rk}\,\mathrm{Idl}(X) < \omega^2$" to the more technical condition "$\mathrm{rk}\,[B] < \omega^2$" may allow one to extend the notion of very-WSTS.

[7] Recursive Petri nets are WSTS for the tree embedding.

We know from [BFP12] that model checking of the *eventually increasing Presburger CTL* fragment of CTL, which has been defined by Atig and Habermehl in [AH11], is undecidable for post-self-modifying nets, while it is decidable for Petri nets [AH11]. However, to the best of our knowledge, the (un)decidability of LTL model checking for post-self-modifying nets is still open. One could hope to show decidability using our framework by proving ideal-increasing-effectiveness.

It also remains to establish the computational complexity of LTL model checking for ω-Petri nets, which cannot directly be done in our framework. It might be possible to adapt extended Rackoff techniques as done for termination in ω-Petri nets [GHPR15].

References

[ACABJ04] Parosh Aziz Abdulla, Aurore Collomb-Annichini, Ahmed Bouajjani, and Bengt Jonsson. Using forward reachability analysis for verification of lossy channel systems. *Formal Methods in System Design*, 25(1):39–65, 2004.

[ACJT96] Parosh Aziz Abdulla, Karlis Cerans, Bengt Jonsson, and Yih-Kuen Tsay. General decidability theorems for infinite-state systems. In *Proc. 11th Annual IEEE Symposium on Logic in Computer Science (LICS)*, pages 313–321, 1996.

[ACJT00] Parosh Aziz Abdulla, Karlis Cerans, Bengt Jonsson, and Yih-Kuen Tsay. Algorithmic analysis of programs with well quasi-ordered domains. *Inf. Comput.*, 160(1-2):109–127, 2000.

[AH11] Mohamed Faouzi Atig and Peter Habermehl. On Yen's path logic for Petri nets. *International Journal of Foundations of Computer Science*, 22(4):783–799, 2011.

[AJ94] Parosh Aziz Abdulla and Bengt Jonsson. Undecidable verification problems for programs with unreliable channels. In *Proc. 21st International Colloquium on Automata, Languages and Programming (ICALP)*, pages 316–327, 1994.

[BBS06] Christel Baier, Nathalie Bertrand, and Philippe Schnoebelen. On computing fixpoints in well-structured regular model checking, with applications to lossy channel systems. In *Proc. 13th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning, (LPAR)*, pages 347–361, 2006.

[BFGL17] Michael Blondin, Alain Finkel, and Jean Goubault-Larrecq. Forward analysis for WSTS, part III: Karp-Miller trees. In *Proc. 37th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, pages 16:1–16:15, 2017.

[BFM17] Michael Blondin, Alain Finkel, and Pierre McKenzie. Well behaved transition systems. *Logical Methods in Computer Science*, 13(3), 2017.

[BFM18] Michael Blondin, Alain Finkel, and Pierre McKenzie. Handling infinitely branching well-structured transition systems. *Information and Computation*, 258:28–49, 2018.

[BFP12] Rémi Bonnet, Alain Finkel, and M. Praveen. Extending the Rackoff technique to affine nets. In *Proc. IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, (FSTTCS)*, pages 301–312, 2012.

[BM07] Aaron R. Bradley and Zohar Manna. *The calculus of computation – Decision procedures with applications to verification*. Springer, 2007.

[Bon75] Robert Bonnet. On the cardinality of the set of initial intervals of a partially ordered set. In *Infinite and finite sets: to Paul Erdős on his 60th birthday*, pages 189–198. North-Holland, 1975.

[BS13] Nathalie Bertrand and Philippe Schnoebelen. Computable fixpoints in well-structured symbolic model checking. *Formal Methods in System Design*, 43(2):233–267, 2013.

[CFS11] Pierre Chambart, Alain Finkel, and Sylvain Schmitz. Forward analysis and model checking for trace bounded WSTS. In *Proc. 32nd International Conference on Applications and Theory of Petri Nets*, 2011.

[CFS16] Pierre Chambart, Alain Finkel, and Sylvain Schmitz. Forward analysis and model checking for trace bounded WSTS. *Theor. Comput. Sci.*, 637:1–29, 2016.

[DFS98] Catherine Dufourd, Alain Finkel, and Philippe Schnoebelen. Reset nets between decidability and undecidability. In *Proc. 25th International Colloquium Automata, Languages and Programming (ICALP)*, pages 103–115, 1998.

[EFM99] Javier Esparza, Alain Finkel, and Richard Mayr. On the verification of broadcast protocols. In *Proc.* 14^{th} *Annual IEEE Symposium on Logic in Computer Science (LICS)*, pages 352–359, 1999.

[EN98] E. Allen Emerson and Kedar S. Namjoshi. On model checking for non-deterministic infinite-state systems. In *Proc.* 13^{th} *IEEE Symposium on Logic in Computer Science (LICS)*, pages 70–80, 1998.

[Esp94] Javier Esparza. On the decidability of model checking for several μ-calculi and Petri nets. In *Proc.* 19^{th} *International Colloquium on Trees in Algebra and Programming (CAAP)*, pages 115–129, 1994.

[ET43] Paul Erdős and Alfred Tarski. On families of mutually exclusive sets. *Annals of Mathematics*, 2(44):315–329, 1943.

[FG09] Alain Finkel and Jean Goubault-Larrecq. Forward analysis for WSTS, part I: Completions. In *STACS'09*, pages 433–444, Freiburg, Germany, 2009. Leibniz-Zentrum für Informatik, Intl. Proc. in Informatics 3.

[FG12] Alain Finkel and Jean Goubault-Larrecq. Forward analysis for WSTS, part II: complete WSTS. *Logical Methods in Computer Science*, 8(3), 2012.

[Fin86] Finkel. *Structuration des systèmes de transitions-applications au contrôle du parallélisme par Files FIFO*. PhD thesis, Université Paris Sud Orsay, 1986.

[Fin87] Alain Finkel. A generalization of the procedure of Karp and Miller to well structured transition systems. In *Proc.* 14^{th} *International Colloquium on Automata, Languages and Programming (ICALP)*, pages 499–508, 1987.

[Fin90] Alain Finkel. Reduction and covering of infinite reachability trees. *Information and Computation*, 89(2):144–179, 1990.

[FM14] Emanuele Frittaion and Alberto Marcone. Reverse mathematics and initial intervals. *Ann. Pure Appl. Logic*, 165(3):858–879, 2014.

[FMP04] Alain Finkel, Pierre McKenzie, and Claudine Picaronny. A well-structured framework for analysing Petri net extensions. *Information and Computation*, 195(1-2):1–29, 2004.

[For10] Thomas Forster. A tutorial on countable ordinals. Available from the Web at `https://www.dpmms.cam.ac.uk/~tf/fundamentalsequence.pdf`, November 2010. Read on Feb. 03, 2017.

[FPS01] Alain Finkel and Philippe Schnoebelen. Well-structured transition systems everywhere! *Theoretical Computer Science*, 256(1-2):63–92, 2001.

[Fra86] Roland Fraïssé. Theory of relations. *Studies in Logic and the Foundations of Mathematics*, 118:1–456, 1986.

[GHPR15] Gilles Geeraerts, Alexander Heußner, M. Praveen, and Jean-François Raskin. ω-Petri nets: Algorithms and complexity. *Fundamenta Informaticae*, 137(1):29–60, 2015.

[Gou13] Jean Goubault-Larrecq. A constructive proof of the topological Kruskal theorem. In *Proc.* 38^{th} *International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 22–41, 2013.

[GRB06] Gilles Geeraerts, Jean-François Raskin, and Laurent Van Begin. Expand, enlarge and check: New algorithms for the coverability problem of WSTS. *Journal of Computer and System Sciences*, 72(1):180–203, 2006.

[Hab97] Peter Habermehl. On the complexity of the linear-time μ-calculus for Petri nets. In *Proc.* 18^{th} *International Conference on Application and Theory of Petri Nets (ICATPN)*, pages 102–116, 1997.

[HLL+16] Piotr Hofman, Slawomir Lasota, Ranko Lazic, Jérôme Leroux, Sylvain Schmitz, and Patrick Totzke. Coverability trees for Petri nets with unordered data. In *Proc.* 19^{th} *International Conference on Foundations of Software Science and Computation Structures (FoSSaCS)*, pages 445–461, 2016.

[HMM14] Reiner Hüchting, Rupak Majumdar, and Roland Meyer. Bounds on mobility. In *Proc.* 25^{th} *International Conference on Concurrency Theory (CONCUR)*, pages 357–371, 2014.

[HP07] Serge Haddad and Denis Poitrenaud. Recursive Petri nets. *Acta Informatica*, 44(7-8):463–508, 2007.

[KM67] Richard M. Karp and Raymond E. Miller. Parallel program schemata: a mathematical model for parallel computation. In *Proc.* 8^{th} *Annual Symposium on Switching and Automata Theory*, pages 55–61. IEEE Computer Society, 1967.

[KP92] M. Kabil and M. Pouzet. Une extension d'un théorème de P. Julien sur les âges de mots. *Informatique théorique et applications*, 26(5):449–482, 1992.

[KSS04] E. V. Kouzmin, Nikolay V. Shilov, and Valery A. Sokolov. Model checking mu-calculus in well-structured transition systems. In *Proc.* 11^{th} *International Symposium on Temporal Representation and Reasoning (TIME)*, pages 152–155, 2004.

[LMP87] J.D. Lawson, M. Mislove, and H. Priestley. Ordered sets with no infinite antichains. *Discrete Mathematics*, 63(2):225–230, 1987.

[Par66] Rohit J. Parikh. On context-free languages. *Journal of the ACM*, 13(4):570–581, 1966.

[Pou79] Maurice Pouzet. Relations non reconstructibles par leurs restrictions. *Journal of Combinatorial Theory, Series B*, 26(1):22–34, 1979.

[Pre29] Mojżesz Presburger. Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. *Comptes Rendus du* 1^{er} *congrès des mathématiciens des pays slaves*, pages 192–201, 1929.

[PZ85] Maurice Pouzet and Nejib Zaguia. Dimension de Krull des ensembles ordonnés. *Discrete Mathematics*, 53:173–192, 1985.

[RM12] Fernando Rosa-Velardo and María Martos-Salgado. Multiset rewriting for the verification of depth-bounded processes with name binding. *Inf. Comput.*, 215:68–87, 2012.

[RMdF11] Fernando Rosa-Velardo, María Martos-Salgado, and David de Frutos-Escrig. Accelerations for the coverability set of Petri nets with names. *Fundamenta Informaticae*, 113(3-4):313–341, 2011.

[Sch10] Philippe Schnoebelen. Lossy counter machines decidability cheat sheet. In *Proc.* 4^{th} *International Workshop on Reachability Problems (RP)*, pages 51–75, 2010.

[Val78] Rüdiger Valk. Self-modifying nets, a natural extension of Petri nets. In *Proc.* 5^{th} *International Colloquium on Automata, Languages and Programming (ICALP)*, pages 464–476, 1978.

[VG05] Kumar N. Verma and Jean Goubault-Larrecq. Karp-Miller trees for a branching extension of VASS. *Discrete Mathematics & Theoretical Computer Science*, 7(1):217–230, 2005.

[VJ85] Rüdiger Valk and Matthias Jantzen. The residue of vector sets with applications to decidability problems in Petri nets. *Acta Informatica*, 21:643–674, 1985.

[Yen92] Hsu-Chun Yen. A unified approach for deciding the existence of certain Petri net paths. *Information and Computation*, 96(1):119–137, 1992.

[ZWH12] Damien Zufferey, Thomas Wies, and Thomas A. Henzinger. Ideal abstractions for well-structured transition systems. In *VMCAI*, pages 445–460, 2012.

8. Appendix

Proof of Proposition 3.4(1–3).

(1) By induction on $|w|$. When $|w| = 0$, the claim is obvious. Otherwise, write w as av where $a \in \Sigma$, $v \in \Sigma^*$, $|v| < |w|$, and let $x \xrightarrow{a} z \xrightarrow{v} y$, for some state z. Certainly z is in $\mathrm{Post}_{\mathcal{S}}(I, a)$, hence in $\downarrow(\mathrm{Post}_{\mathcal{S}}(I, a))$. Write the ideal decomposition of the latter as $\{I_1, I_2, \ldots, I_n\}$. For some k, $1 \leq k \leq n$, z is in I_k, and by definition $I \overset{a}{\rightsquigarrow} I_k$. By induction hypothesis, $I_k \overset{v}{\rightsquigarrow} J$ for some ideal containing y, whence the result.

(2) By induction of $|w|$ again. The case $|w| = 0$ is obvious, too. Otherwise, write w as av, where $a \in \Sigma$, $v \in \Sigma^*$, $|v| < |w|$. There is an ideal K such that $I \overset{a}{\rightsquigarrow} K \overset{v}{\rightsquigarrow} J$, and the induction hypothesis gives us elements $z \in K$ and $y' \in J$, and a word $v' \in \Sigma^*$ such that $z \xrightarrow{v'} y'$ and $y' \geq y$. (Moreover, if $\mathcal{S}$ has strong monotonicity, then $v' = v$.) By definition of $\overset{a}{\rightsquigarrow}$, K is included in $\downarrow \mathrm{Post}_{\mathcal{S}}(I, a)$, so there are elements $x \in I$ and $z' \in K$ with $z' \geq z$ such that $x \xrightarrow{a} z'$. Since $\mathcal{S}$ is monotonic, there is a further element $y'' \geq y'$ and a further word v'' such that $z' \xrightarrow{v''} y''$. (If $\mathcal{S}$ is strongly monotonic, $v'' = v'$, so $v'' = v$.) This entails that $x \xrightarrow{av''} y'' \geq y$, and if $\mathcal{S}$ is strongly monotonic, $av'' = av = w$.

(3) Let $J \in \mathrm{Post}_{\widehat{\mathcal{S}}}(I, w)$ and let $y \in J$. By (2), there exist $x \in I$ and $y' \in X$ such that $x \xrightarrow{w} y'$ and $y' \geq y$. Thus, $y \in \downarrow \mathrm{Post}_{\mathcal{S}}(x, w) \subseteq \downarrow \mathrm{Post}_{\mathcal{S}}(I, w)$. Conversely, let $y \in \downarrow \mathrm{Post}_{\mathcal{S}}(I, w)$. There exist $x \in I$ and $y' \in X$ such that $x \xrightarrow{w} y'$ and $y' \geq y$. By (1), there exists an ideal $J \supseteq \downarrow y' \supseteq \downarrow y$ such that $I \overset{w}{\rightsquigarrow} J$. Thus, $J \in \mathrm{Post}_{\widehat{\mathcal{S}}}(I, w)$ and $y \in J$. □