Formal analysis of crowd systems

Michael Blondin

Université de Sherbrooke

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Joint work with J. Esparza, M. Helfrich, S. Jaax, A. Kučera, P. J. Meyer

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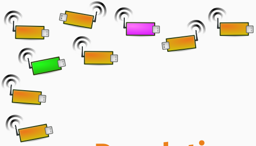
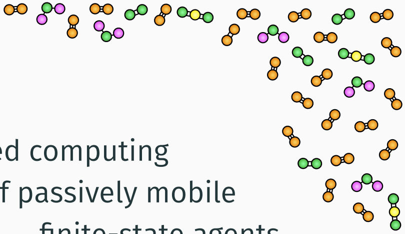
Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

Overview





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Model *e.g.* networks of passively mobile sensors and chemical reaction networks

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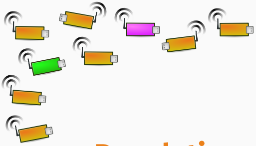

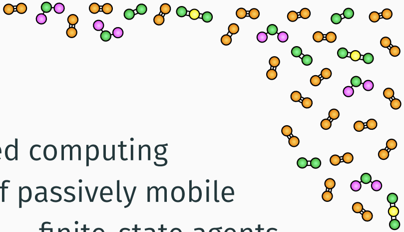

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

Model *e.g.* networks of passively mobile sensors and chemical reaction networks

Protocols compute predicates of the form $\varphi \colon \mathbb{N}^d \to \{0, 1\}$

e.g. $\varphi(m, n)$ is computed by $m + n$ agents

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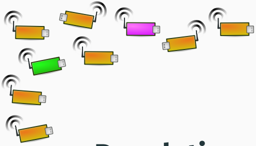

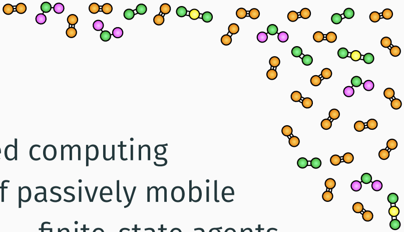

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

This talk: automatic verification and expected termination time analysis

Population protocols

- anonymous mobile agents with very few resources
- agents change states via random pairwise interactions
- each agent has opinion true/false
- computes by stabilizing agents to some opinion

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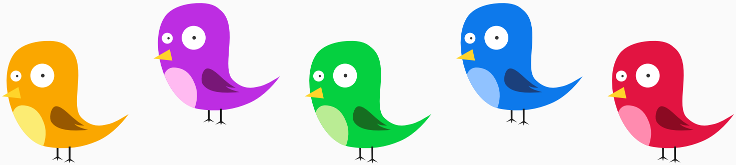

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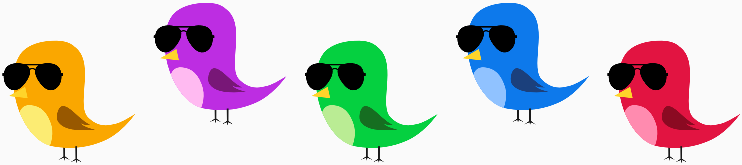

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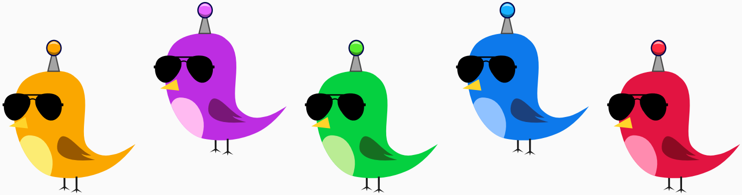

Population protocols

Angluin *et al.* PODC'04

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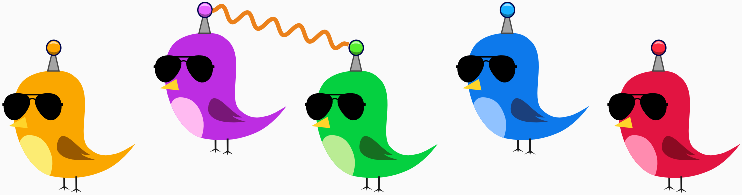

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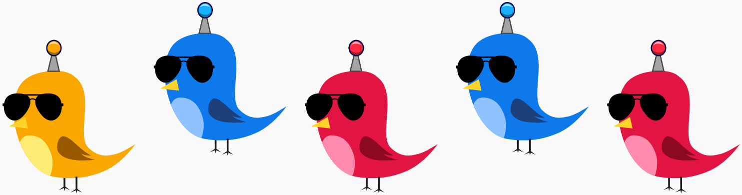

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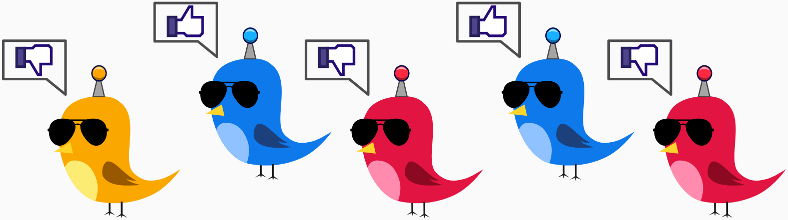

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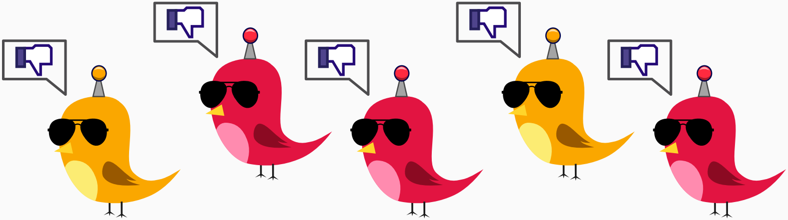

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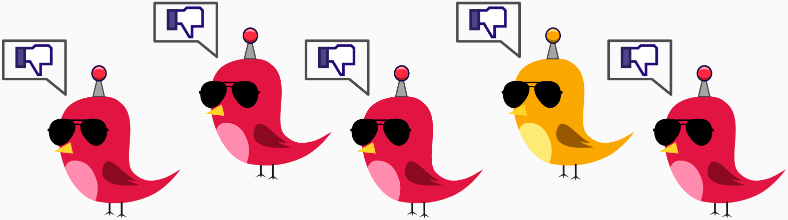

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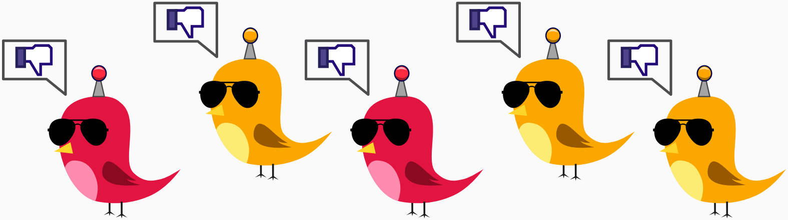
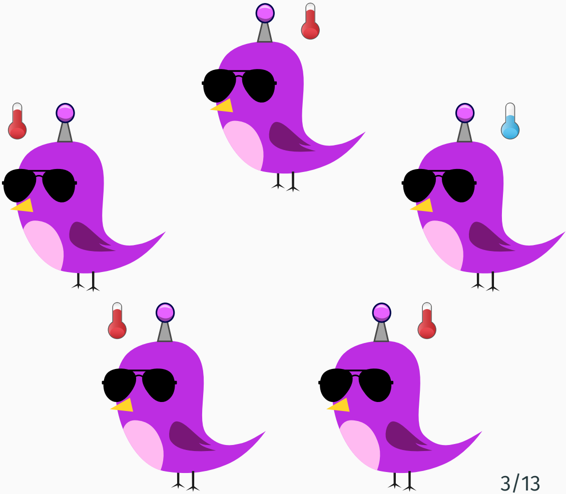


Are there at least 4 sick birds?



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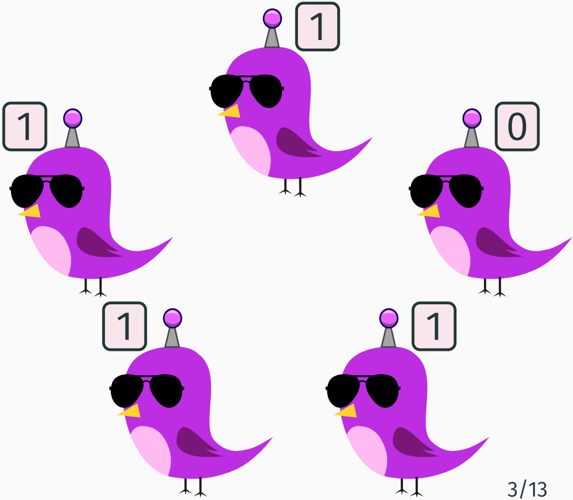

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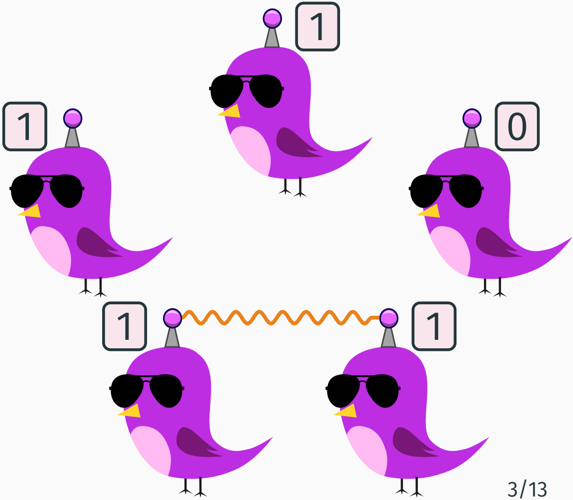

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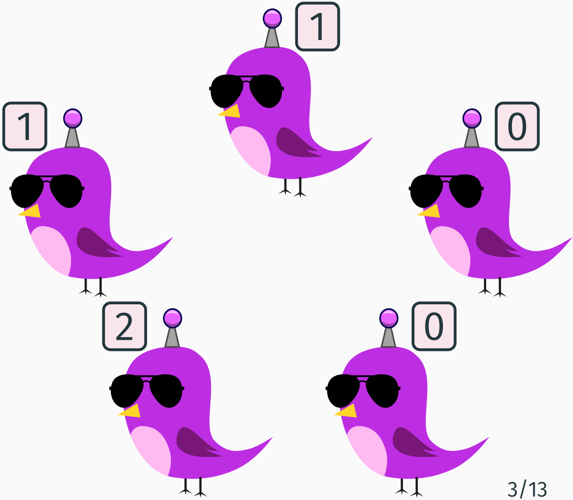

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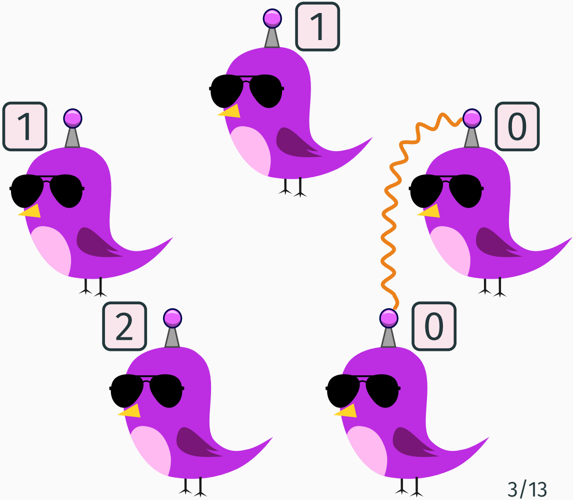

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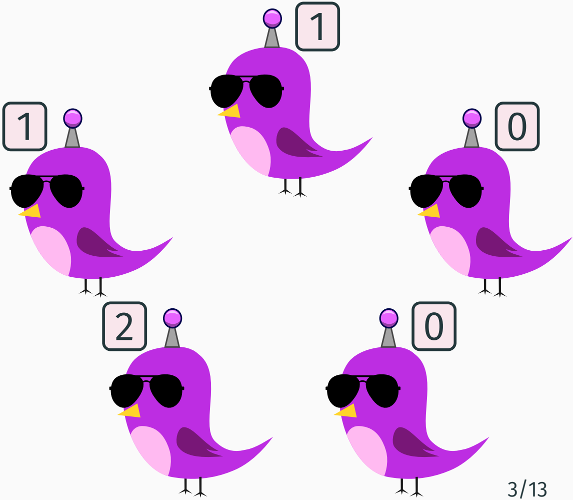

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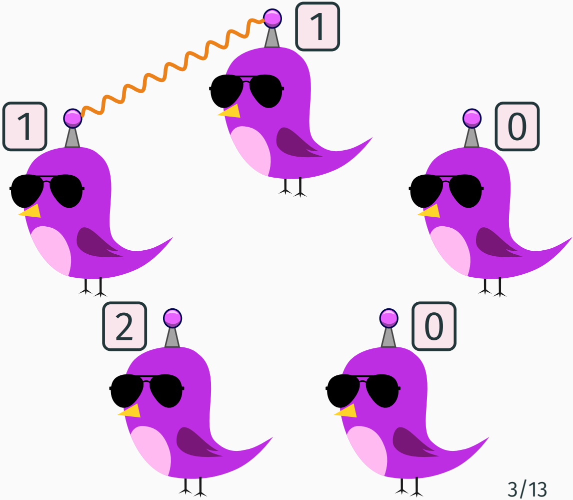

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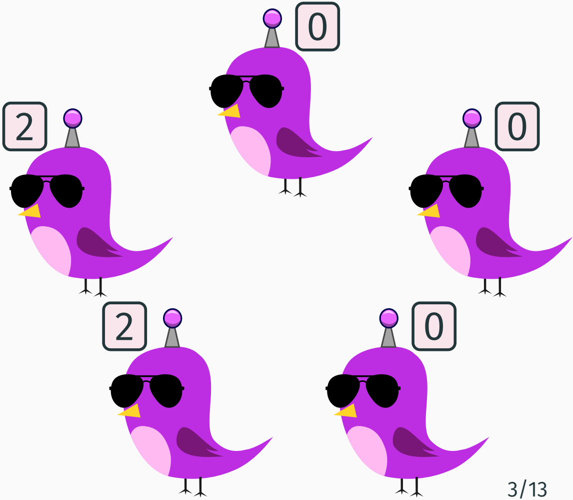

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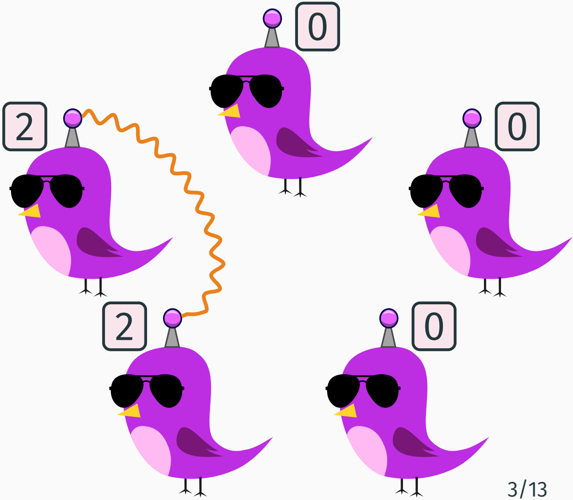

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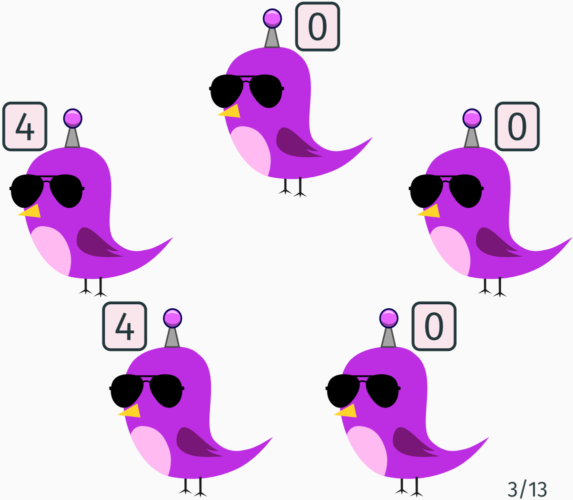

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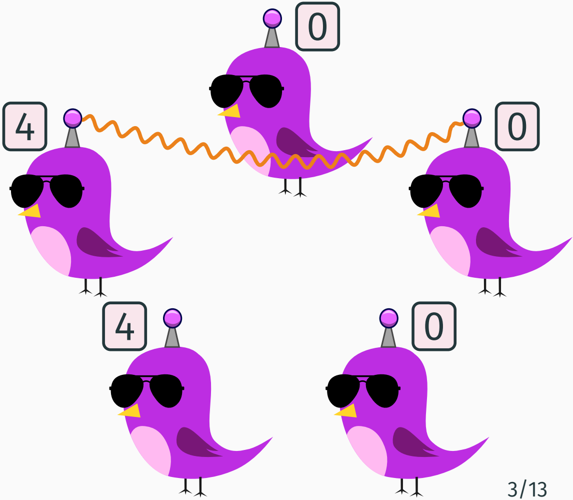

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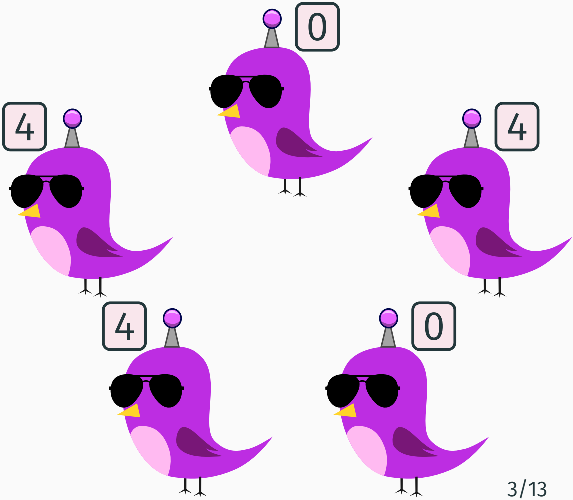

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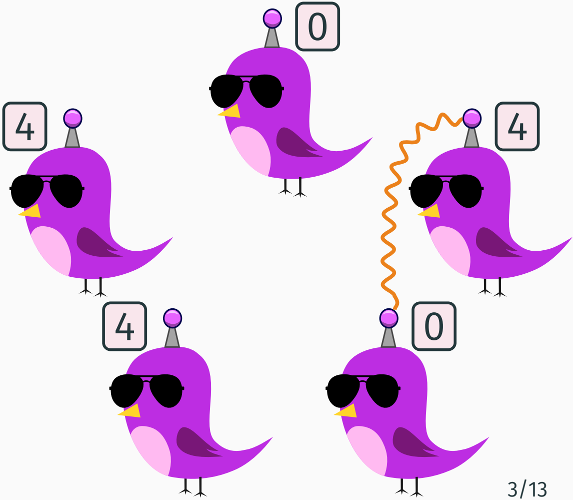

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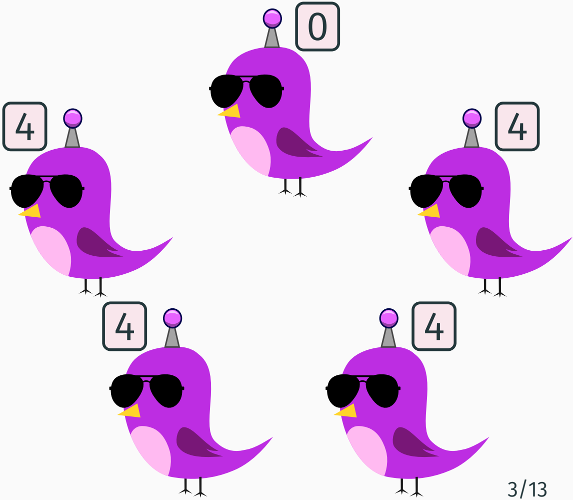

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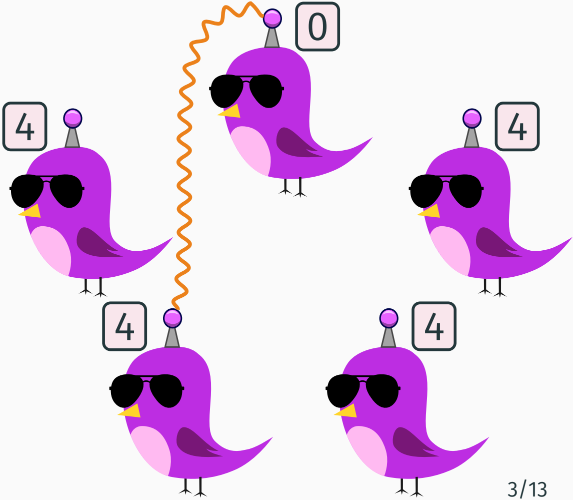

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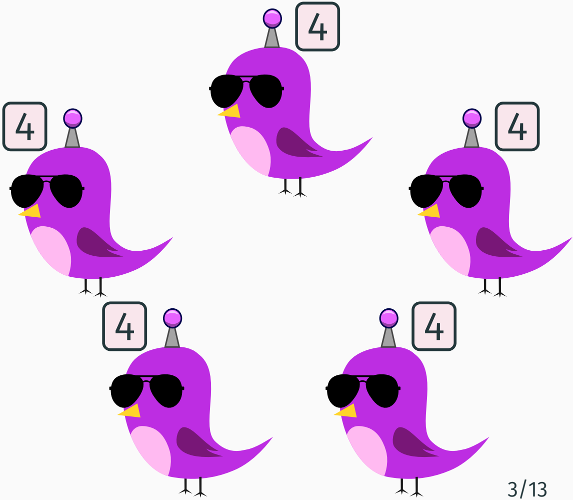

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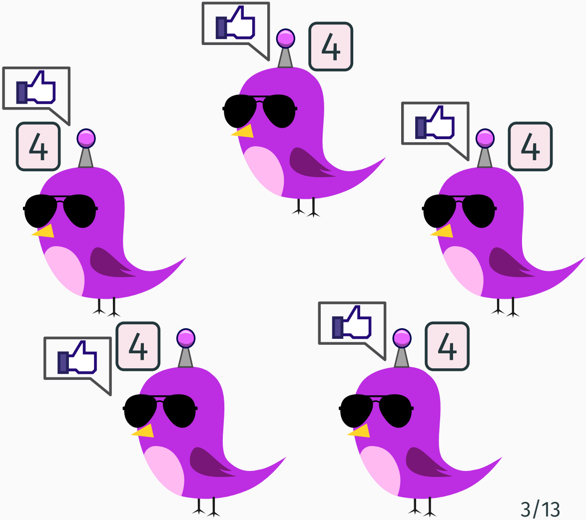
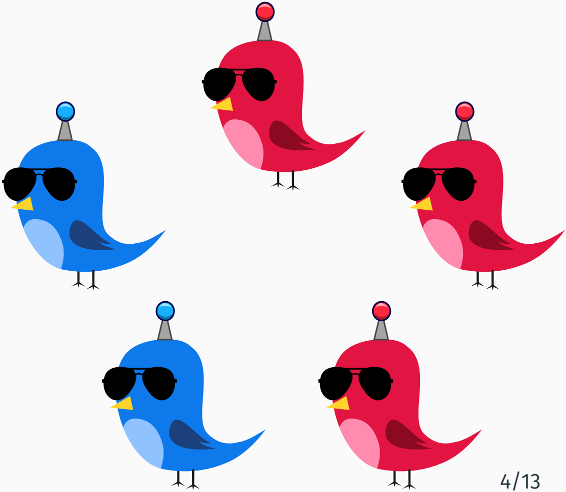


Example: majority protocol

blue agents ≥ # red agents?



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- Two large agents become small blue agents
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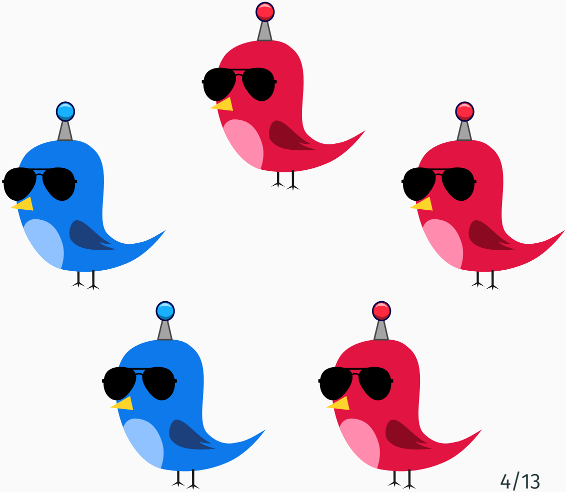

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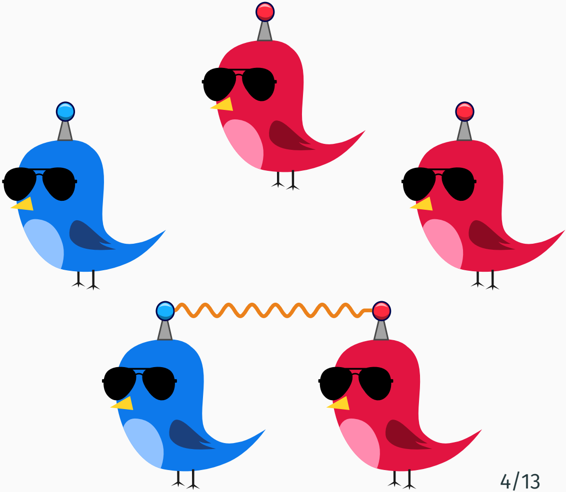

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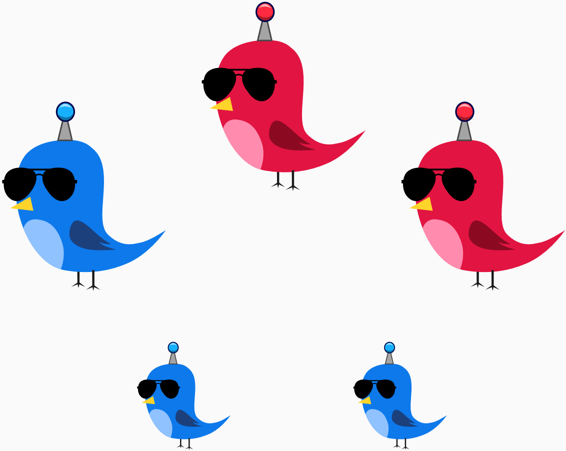

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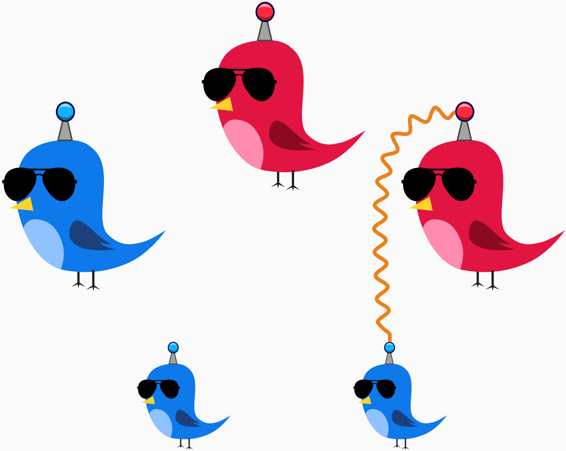

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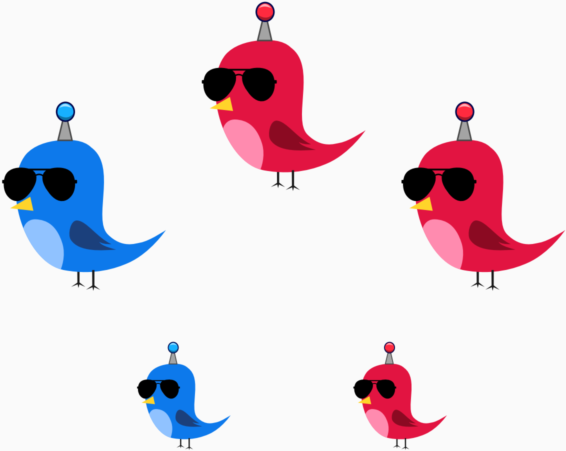

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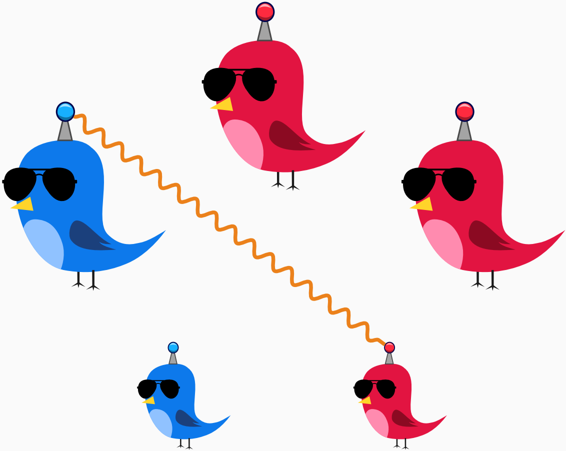

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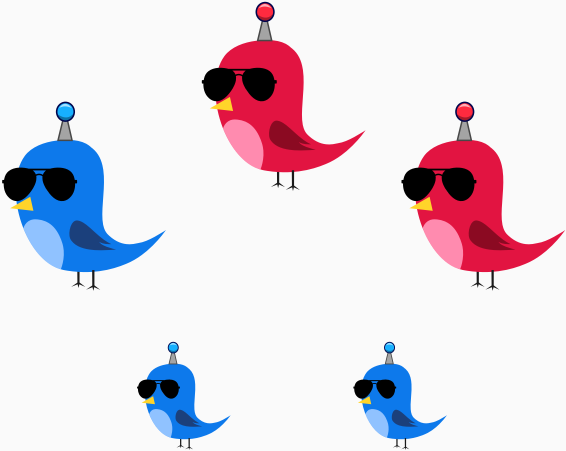

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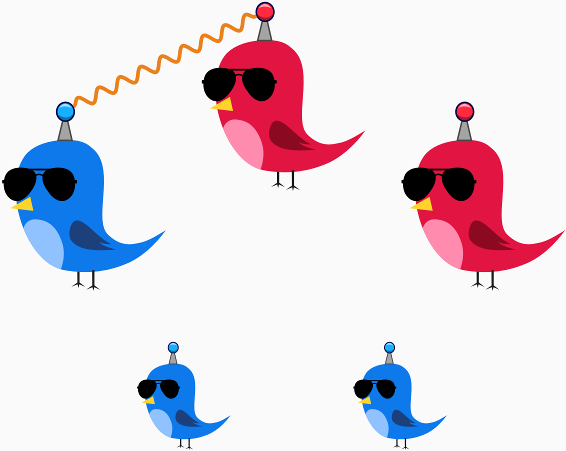

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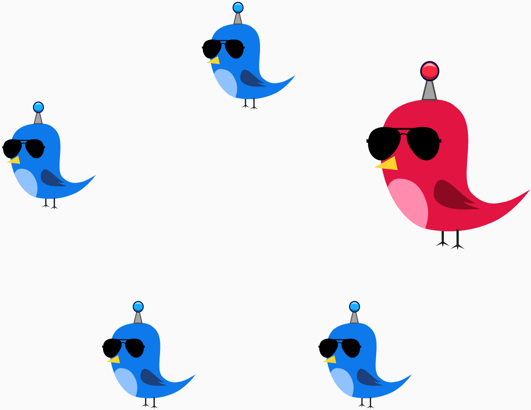

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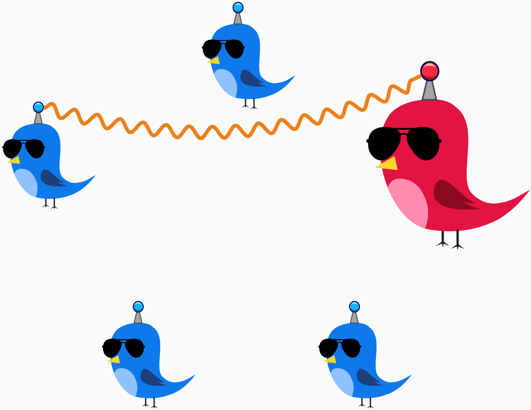

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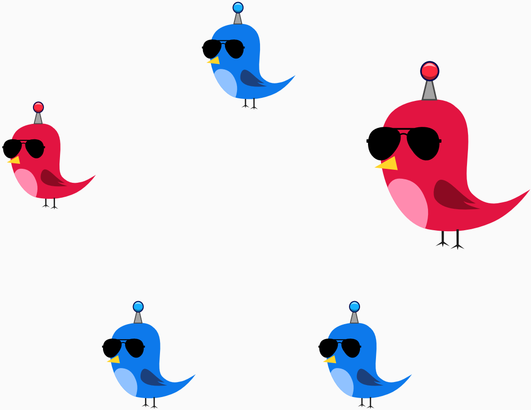

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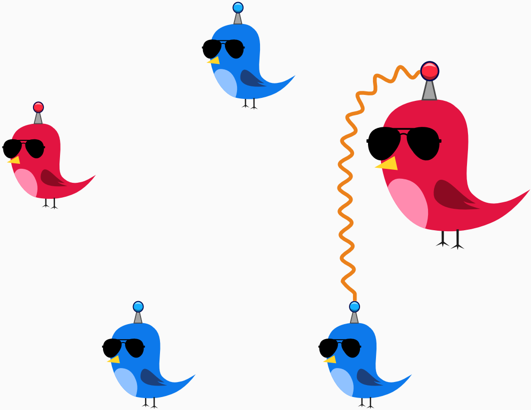

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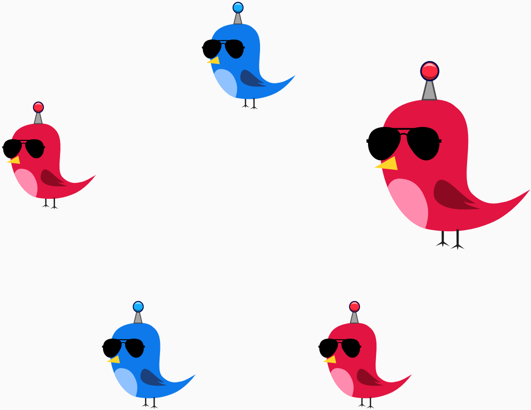

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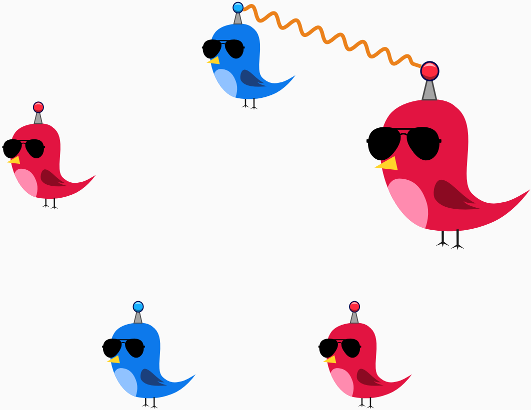

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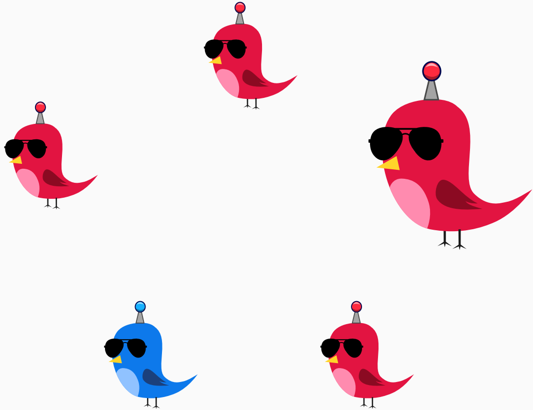

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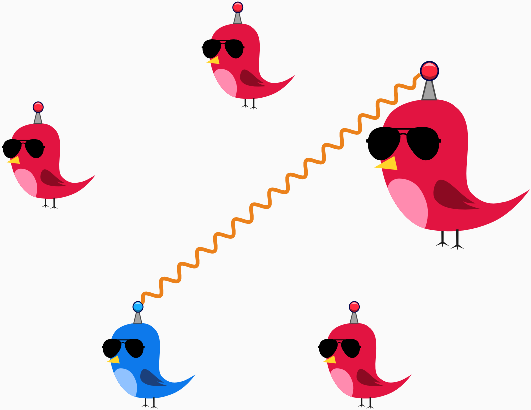

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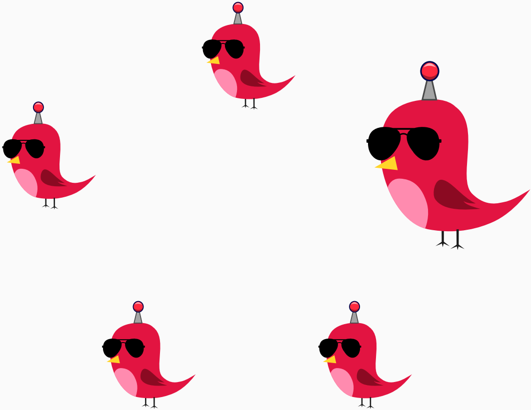

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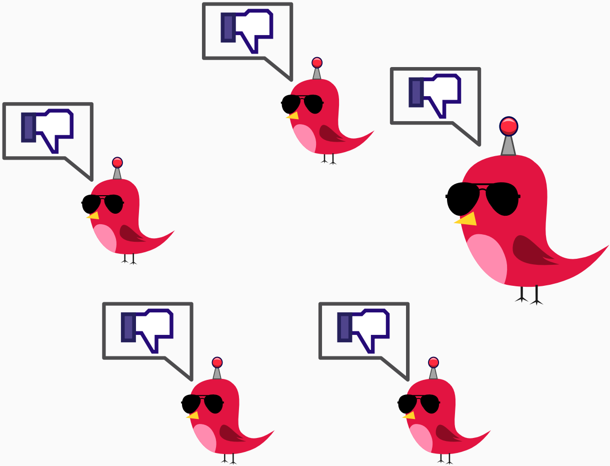
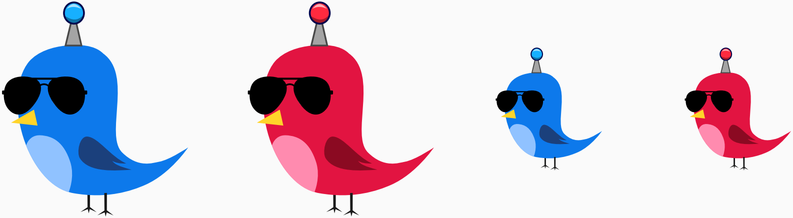


Demonstration

Population protocols: formal model

- *States*: finite set Q
- *Opinions*: $O\colon Q \to \{\mathtt{false}, \mathtt{true}\}$
- *Initial states*: $I \subseteq Q$
- *Transitions*: $T \subseteq Q^2 \times Q^2$



Population protocols: formal model

- *States*: finite set Q
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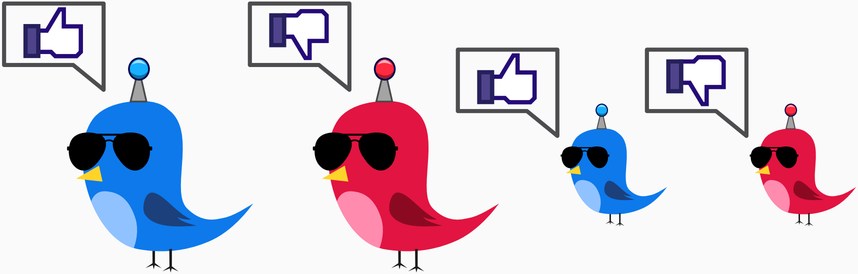

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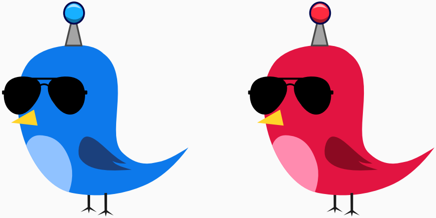

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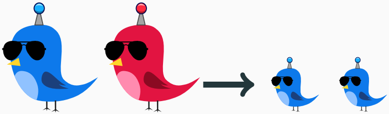
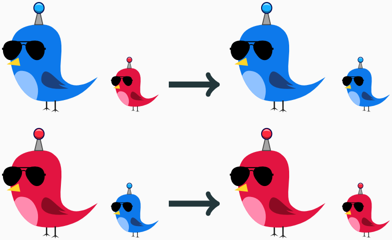
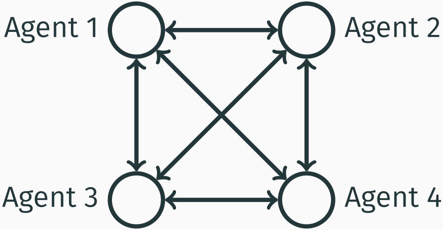




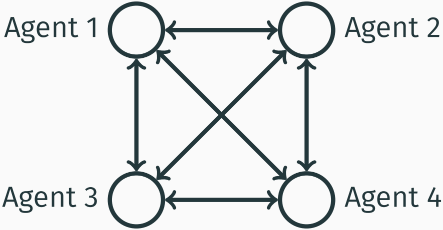
All agents can interact pairwise

(complete topology)



Population protocols: interactions

$$\mathbb{P}[\text{fire } p, q \mapsto p', q' \text{ in } C] = \begin{cases} \dfrac{2 \cdot C(p) \cdot C(q)}{n^2 - n} & \text{if } p \neq q \\[2ex] \dfrac{C(p) \cdot (C(p) - 1)}{n^2 - n} & \text{if } p = q \end{cases}$$



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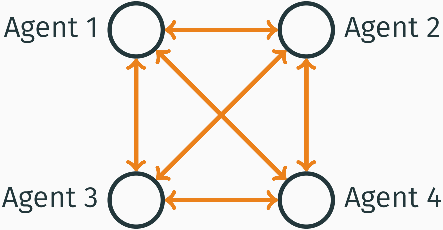

Population protocols: interactions

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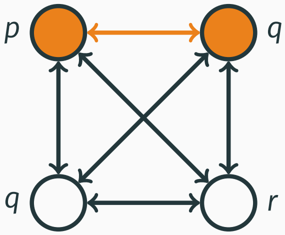

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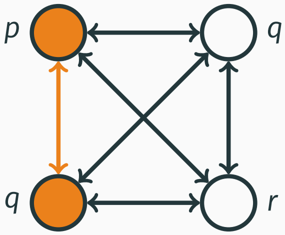
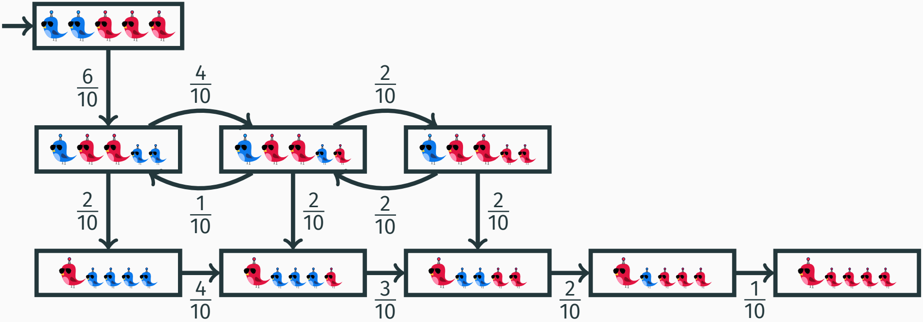


Population protocols: interactions

$$\mathbb{P}[\text{fire } p, q \mapsto p', q' \text{ in } C] \;=\; \begin{cases} \dfrac{2 \cdot C(p) \cdot C(q)}{n^2 - n} & \text{if } p \neq q \\[2ex] \dfrac{C(p) \cdot (C(p) - 1)}{n^2 - n} & \text{if } p = q \end{cases}$$

$$\mathbb{P}[C \to C'] \;=\; \sum_{t \text{ s.t. } C \xrightarrow{t} C'} \mathbb{P}[\text{fire } t \text{ in } C]$$

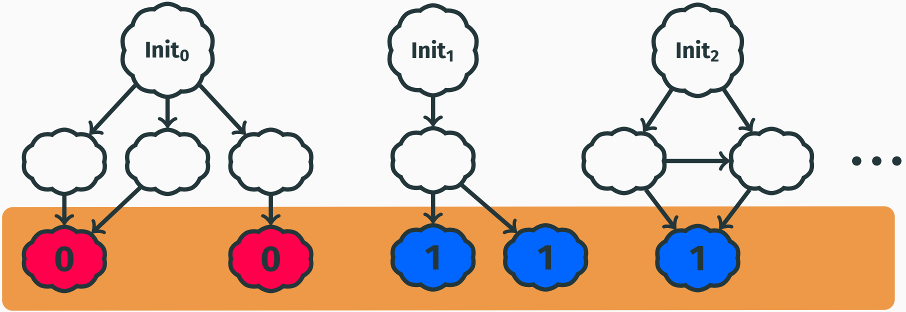
Underlying Markov chain:



A protocol computes a predicate $f\colon \mathbb{N}^I \to \{0,1\}$

if runs reach **common stable consensus**

with probability 1



A protocol computes a predicate $f\colon \mathbb{N}^I \to \{0,1\}$

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with probability 1

Expressive power **Angluin, Aspnes, Eisenstat PODC'06**

Population protocols compute precisely predicates definable in Presburger arithmetic, *i.e.* FO($\mathbb{N}$, $+$, $<$)

Verifying correctness

Protocol broken for B = R:

$$B\ R \to b\ b$$
$$B\ r \to B\ b$$
$$R\ b \to R\ r$$

Verifying correctness

Protocol broken for B = R:

$$\boldsymbol{B}\ \boldsymbol{R} \to \boldsymbol{b}\ \boldsymbol{b}$$
$$\boldsymbol{B}\ \boldsymbol{r} \to \boldsymbol{B}\ \boldsymbol{b}$$
$$\boldsymbol{R}\ \boldsymbol{b} \to \boldsymbol{R}\ \boldsymbol{r}$$

B R B R

Verifying correctness

Protocol broken for B = R:

$$B\ R \rightarrow b\ b$$
$$B\ r \rightarrow B\ b$$
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B R B R → B R b b

Verifying correctness

Protocol broken for B = R:

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$$\boldsymbol{B}\ \boldsymbol{r} \rightarrow \boldsymbol{B}\ \boldsymbol{b}$$
$$\boldsymbol{R}\ \boldsymbol{b} \rightarrow \boldsymbol{R}\ \boldsymbol{r}$$

$$\mathbf{B\ R\ B\ R} \rightarrow \mathbf{B\ R\ b\ b} \rightarrow \mathbf{B\ R\ r\ b}$$

Verifying correctness

Protocol broken for B = R:

$$B\ R \rightarrow b\ b$$
$$B\ r \rightarrow B\ b$$
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$$\mathbf{B\ R\ B\ R} \rightarrow \mathbf{B\ R\ b\ b} \rightarrow \mathbf{B\ R\ r\ b} \rightarrow \mathbf{b\ b\ r\ b}$$

Verifying correctness

Protocol correct with tie-breaker:

$$\begin{aligned} B\,R &\rightarrow b\,b \\ B\,r &\rightarrow B\,b \\ R\,b &\rightarrow R\,r \\ b\,r &\rightarrow b\,b \end{aligned}$$

$$\mathbf{B\,R\,B\,R} \rightarrow \mathbf{B\,R\,b\,b} \rightarrow \mathbf{B\,R\,r\,b} \rightarrow \mathbf{b\,b\,r\,b}$$

Verifying correctness

Protocol correct with tie-breaker:

$$\begin{aligned} \boldsymbol{B}\,\boldsymbol{R} &\to \boldsymbol{b}\,\boldsymbol{b} \\ \boldsymbol{B}\,\boldsymbol{r} &\to \boldsymbol{B}\,\boldsymbol{b} \\ \boldsymbol{R}\,\boldsymbol{b} &\to \boldsymbol{R}\,\boldsymbol{r} \\ \boldsymbol{b}\,\boldsymbol{r} &\to \boldsymbol{b}\,\boldsymbol{b} \end{aligned}$$

B R B R $\to$ **B R b b** $\to$ **B R r b** $\to$ **b b r b** $\to$ **b b b b**

Easy fix, but protocols can become complex even for B ≥ R:

Fast and Exact Majority in Population Protocols

Dan Alistarh
Microsoft Research

Rati Gelashvili*
MIT

Milan Vojnović
Microsoft Research

$$weight(x) = \begin{cases} |x| & \text{if } x \in StrongStates \text{ or } x \in WeakStates; \\ 1 & \text{if } x \in IntermediateStates. \end{cases}$$

$$sgn(x) = \begin{cases} 1 & \text{if } x \in \{+0, 1_d, \ldots, 1_1, 3, 5, \ldots, m\}; \\ -1 & \text{otherwise.} \end{cases}$$

$value(x) = sgn(x) \cdot weight(x)$

`/* Functions for rounding state interactions */`

$\phi(x) = -1_1$ if $x = -1$; 1_1 if $x = 1$; x, otherwise

$R_{\downarrow}(k) = \phi(k$ if k odd integer, $k-1$ if k even)

$R_{\uparrow}(k) = \phi(k$ if k odd integer, $k+1$ if k even)

$$Shift\text{-}to\text{-}Zero(x) = \begin{cases} -1_{j+1} & \text{if } x = -1_j \text{ for some index } j < d \\ 1_{j+1} & \text{if } x = 1_j \text{ for some index } j < d \\ x & \text{otherwise.} \end{cases}$$

$$Sign\text{-}to\text{-}Zero(x) = \begin{cases} +0 & \text{if } sgn(x) > 0 \\ -0 & \text{oherwise.} \end{cases}$$

procedure update$\langle x, y\rangle$
  if $(weight(x) > 0$ *and* $weight(y) > 1)$ ***or*** $(weight(y) > 0$ *and* $weight(x) > 1)$ **then**
    $x' \leftarrow R_{\downarrow}\left(\frac{value(x)+value(y)}{2}\right)$ and $y' \leftarrow R_{\uparrow}\left(\frac{value(x)+value(y)}{2}\right)$
  else if $weight(x) \cdot weight(y) = 0$ *and* $value(x) + value(y) > 0$ **then**
    if $weight(x) \neq 0$ **then** $x' \leftarrow Shift\text{-}to\text{-}Zero(x)$ and $y' \leftarrow Sign\text{-}to\text{-}Zero(x)$
    else $y' \leftarrow Shift\text{-}to\text{-}Zero(y)$ and $x' \leftarrow Sign\text{-}to\text{-}Zero(y)$
  else if $(x \in \{-1_d, +1_d\}$ *and* $weight(y) = 1$ *and* $sgn(x) \neq sgn(y))$ ***or***
      $(y \in \{-1_d, +1_d\}$ *and* $weight(x) = 1$ *and* $sgn(y) \neq sgn(x))$ **then**
    $x' \leftarrow -0$ and $y' \leftarrow +0$
  else
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Verifying correctness

Easy fix, but protocols can become complex even for B ≥ R:

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```
/* Functions for rounding state interactions */
```

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    $x' \leftarrow -0$ and $y' \leftarrow +0$
  else
    $x' \leftarrow Shift\text{-}to\text{-}Zero(x)$ and $y' \leftarrow Shift\text{-}to\text{-}Zero(y)$

How to verify correctness automatically?

Testing whether a protocol computes φ amounts to testing:

$$\begin{aligned}\neg\exists C, D:\quad & C \xrightarrow{*} D\ \wedge \\ & C \text{ is initial} \wedge \\ & D \text{ is in a BSCC} \wedge \\ & \text{opinion}(D) \neq \varphi(C)\end{aligned}$$

Verifying correctness

Testing whether a protocol computes φ amounts to testing:

$$\begin{aligned} \neg\exists C, D:\quad & C \xrightarrow{*} D \,\wedge \\ & C \text{ is initial} \,\wedge \\ & D \text{ is in a BSCC} \,\wedge \\ & \text{opinion}(D) \neq \varphi(C) \end{aligned}$$

Theorem **Esparza *et al.* CONCUR'15**

Verification is decidable

Verification: 1st approach

$$\neg\exists C, D: \quad C \xrightarrow{*} D \wedge$$
$$C \text{ is initial} \wedge$$
$$D \text{ is in a BSCC} \wedge$$
$$\text{opinion}(D) \neq \varphi(C)$$

As difficult as verification

Ackermannan-complete

(Leroux; Czerwinski & Orlikowski FOCS'21, Esparza et al. CONCUR'15)

$$\begin{aligned} \neg\exists C, D:\quad & C \xrightarrow{*} D \wedge \\ & C \text{ is initial } \wedge \\ & D \text{ is in a BSCC } \wedge \\ & \text{opinion}(D) \neq \varphi(C) \end{aligned}$$

Relaxed with Presburger-definable overapproximation!

Verification: 1st approach

$$\neg\exists C, D: \quad C \overset{*}{\dashrightarrow} D \wedge$$

$$C \text{ is initial} \wedge$$

$$D \text{ is in a BSCC} \wedge$$

$$\text{opinion}(D) \neq \varphi(C)$$

Difficult to express

$$\neg\exists C, D: \quad C \overset{*}{\dashrightarrow} D \wedge$$
$$C \text{ is initial} \wedge$$
$$D \text{ is terminal} \wedge$$
$$\text{opinion}(D) \neq \varphi(C)$$

BSCCs are of size 1
for many protocols!

Verification: 1st approach

$$\neg\exists C, D:\quad \begin{aligned} & C \overset{*}{\dashrightarrow} D \,\wedge \\ & C \text{ is initial} \wedge \\ & D \text{ is terminal } \wedge \\ & \text{opinion}(D) \neq \varphi(C) \end{aligned}$$

Testable with an SMT solver

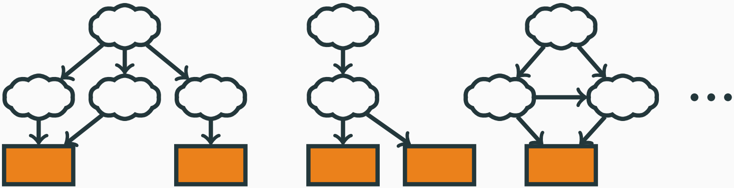
Verification: 1st approach

$$\begin{aligned} \neg\exists C, D:\ & C \overset{*}{\dashrightarrow} D \wedge \\ & C \text{ is initial} \wedge \\ & D \text{ is terminal} \wedge \\ & \text{opinion}(D) \neq \varphi(C) \end{aligned}$$

But how to know whether all BSCCs are of size 1?

Silent protocols

A protocol is *silent* if fair executions reach terminal configurations

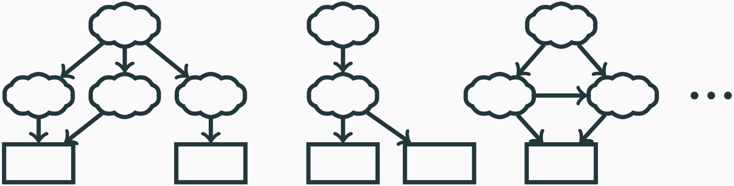


BSCCs of size 1

Silent protocols

A protocol is *silent* if fair executions reach terminal configurations

- Testing silentness is as hard as verification of correctness
- But many protocols satisfy a common design

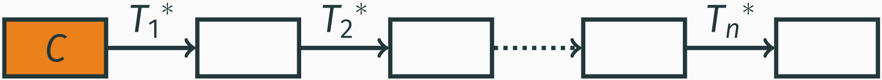


BSCCs of size 1

Silent protocols: layered termination

Partition $T = T_1 \cup T_2 \cup \cdots \cup T_n$ s.t. for every i

- all executions restricted to T_i terminate
- if $T_1 \cup \cdots \cup T_{i-1}$ disabled in C and $C \xrightarrow{T_i{}^*} D$, then $T_1 \cup \cdots \cup T_{i-1}$ also disabled in D



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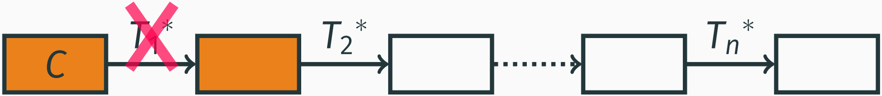

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Silent protocols: layered termination

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Silent protocols: layered termination

T_1

$$\begin{aligned} B\ R &\to b\ b \\ B\ r &\to B\ b \\ R\ b &\to R\ r \\ b\ r &\to b\ b \end{aligned}$$

Silent protocols: layered termination

T_1

$$\begin{aligned} \boldsymbol{B}\ \boldsymbol{R} &\rightarrow \boldsymbol{b}\ \boldsymbol{b} \\ \boldsymbol{B}\ \boldsymbol{r} &\rightarrow \boldsymbol{B}\ \boldsymbol{b} \\ \boldsymbol{R}\ \boldsymbol{b} &\rightarrow \boldsymbol{R}\ \boldsymbol{r} \\ \boldsymbol{b}\ \boldsymbol{r} &\rightarrow \boldsymbol{b}\ \boldsymbol{b} \end{aligned}$$

Bad partition: not all executions over T_1 terminate

Silent protocols: layered termination

T_1

$$\begin{aligned} B\ R &\to b\ b \\ \boxed{B\ r} &\to \boxed{B\ b} \\ \boxed{R\ b} &\to \boxed{R\ r} \\ b\ r &\to b\ b \end{aligned}$$

Bad partition: not all executions over T_1 terminate

$$\{B, B, R, R\} \to \{B, b, b, R\} \to \{B, b, r, R\} \to \{B, b, b, R\} \to \{B, b, r, R\} \to \cdots$$

Silent protocols: layered termination

T_1

$$\boldsymbol{B}\ \boldsymbol{R} \rightarrow \boldsymbol{b}\ \boldsymbol{b}$$

T_2

$$\boldsymbol{R}\ \boldsymbol{b} \rightarrow \boldsymbol{R}\ \boldsymbol{r}$$

T_3

$$\boldsymbol{B}\ \boldsymbol{r} \rightarrow \boldsymbol{B}\ \boldsymbol{b}$$

$$\boldsymbol{b}\ \boldsymbol{r} \rightarrow \boldsymbol{b}\ \boldsymbol{b}$$

Silent protocols: layered termination

T_1

$$\boldsymbol{B}\ \boldsymbol{R} \rightarrow \boldsymbol{b}\ \boldsymbol{b}$$

T_2

$$\boldsymbol{R}\ \boldsymbol{b} \rightarrow \boldsymbol{R}\ \boldsymbol{r}$$

T_3

$$\boldsymbol{B}\ \boldsymbol{r} \rightarrow \boldsymbol{B}\ \boldsymbol{b}$$

$$\boldsymbol{b}\ \boldsymbol{r} \rightarrow \boldsymbol{b}\ \boldsymbol{b}$$

$\#\boldsymbol{B} \geq \#\boldsymbol{R}$:

$$\{\boldsymbol{B}^*,\ \boldsymbol{R}^*\}$$

Silent protocols: layered termination

T_1 ✗

$$B\ R \to b\ b$$

T_2

$$R\ b \to R\ r$$

T_3

$$B\ r \to B\ b$$

$$b\ r \to b\ b$$

$\#B \geq \#R$:

$$\{B^*,\ R^*\} \xrightarrow{*} \{B^*,\ b^*,\ r^*\}$$

Silent protocols: layered termination

T_1 ✗

$$B\ R \rightarrow b\ b$$

T_2 ✗

$$R\ b \rightarrow R\ r$$

T_3

$$B\ r \rightarrow B\ b$$

$$b\ r \rightarrow b\ b$$

$\#B \geq \#R$:

$$\{B^*,\ R^*\} \xrightarrow{*} \{B^*,\ b^*,\ r^*\}$$

Silent protocols: layered termination

T_1 ✗

$$B\ R \rightarrow b\ b$$

T_2 ✗

$$R\ b \rightarrow R\ r$$

T_3 ✗

$$B\ r \rightarrow B\ b$$
$$b\ r \rightarrow b\ b$$

$\#B \geq \#R$:

$$\{B^*,\ R^*\} \xrightarrow{*} \{B^*,\ b^*,\ r^*\} \xrightarrow{*} \{B^*,\ b^*\}$$

Silent protocols: layered termination

T_1

$$B\ R \to b\ b$$

T_2

$$R\ b \to R\ r$$

T_3

$$B\ r \to B\ b$$
$$b\ r \to b\ b$$

$\#B \geq \#R$:

$$\{B^*,\ R^*\} \xrightarrow{*} \{B^*,\ b^*,\ r^*\} \xrightarrow{*} \{B^*,\ b^*\}$$

$\#R > \#B$:

$$\{R^+,\ B^*\}$$

Silent protocols: layered termination

T_1 ✗

$$\boldsymbol{B}\ \boldsymbol{R} \to \boldsymbol{b}\ \boldsymbol{b}$$

T_2

$$\boldsymbol{R}\ \boldsymbol{b} \to \boldsymbol{R}\ \boldsymbol{r}$$

T_3

$$\boldsymbol{B}\ \boldsymbol{r} \to \boldsymbol{B}\ \boldsymbol{b}$$

$$\boldsymbol{b}\ \boldsymbol{r} \to \boldsymbol{b}\ \boldsymbol{b}$$

$\#\boldsymbol{B} \geq \#\boldsymbol{R}$:

$$\{\boldsymbol{B}^*,\ \boldsymbol{R}^*\} \xrightarrow{*} \{\boldsymbol{B}^*,\ \boldsymbol{b}^*,\ \boldsymbol{r}^*\} \xrightarrow{*} \{\boldsymbol{B}^*,\ \boldsymbol{b}^*\}$$

$\#\boldsymbol{R} > \#\boldsymbol{B}$:

$$\{\boldsymbol{R}^+,\ \boldsymbol{B}^*\} \xrightarrow{*} \{\boldsymbol{R}^+,\ \boldsymbol{b}^*, \boldsymbol{r}^*\}$$

Silent protocols: layered termination

T_1 ✗

$$B\ R \to b\ b$$

T_2 ✗

$$R\ b \to R\ r$$

T_3

$$B\ r \to B\ b$$
$$b\ r \to b\ b$$

$\#B \geq \#R$:

$$\{B^*,\ R^*\} \xrightarrow{*} \{B^*,\ b^*,\ r^*\} \xrightarrow{*} \{B^*,\ b^*\}$$

$\#R > \#B$:

$$\{R^+,\ B^*\} \xrightarrow{*} \{R^+,\ b^*, r^*\} \xrightarrow{*} \{R^+,\ r^*\}$$

Silent protocols: layered termination

T_1 ✗

$B\ R \to b\ b$

T_2 ✗

$R\ b \to R\ r$

T_3 ✗

$B\ r \to B\ b$

$b\ r \to b\ b$

$\#B \geq \#R$:

$$\{B^*,\ R^*\} \xrightarrow{*} \{B^*,\ b^*,\ r^*\} \xrightarrow{*} \{B^*,\ b^*\}$$

$\#R > \#B$:

$$\{R^+,\ B^*\} \xrightarrow{*} \{R^+,\ b^*, r^*\} \xrightarrow{*} \{R^+,\ r^*\}$$

Silent protocols: layered termination

T_1

$$\mathbf{B}\,\mathbf{R} \to \mathbf{b}\,\mathbf{b}$$

T_2

$$\mathbf{R}\,\mathbf{b} \to \mathbf{R}\,\mathbf{r}$$

T_3

$$\mathbf{B}\,\mathbf{r} \to \mathbf{B}\,\mathbf{b}$$

$$\mathbf{b}\,\mathbf{r} \to \mathbf{b}\,\mathbf{b}$$

Theorem

Deciding whether a protocol is strongly silent $\in$ NP

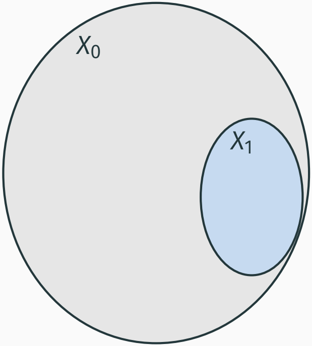
Recent efficient protocols are not silent!

Recent efficient protocols are not silent!

More powerful approach:
using “correctness certificates”

Correctness certificates

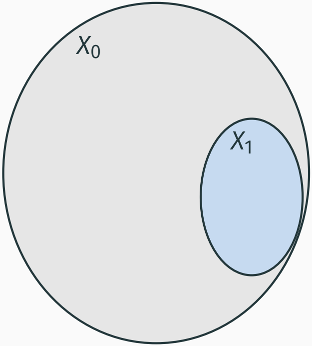
Approach: certify that φ is computed correctly for $b \in \{0, 1\}$



$r_0 \colon \text{Configs} \to \mathbb{N}$

Correctness certificates

Approach: certify that φ is computed correctly for $b \in \{0, 1\}$



r_0 : Configs $\to \mathbb{N}$

- $C \in X_i \land C \xrightarrow{*} C' \implies C' \in X_i$

Closed under reachability

Correctness certificates

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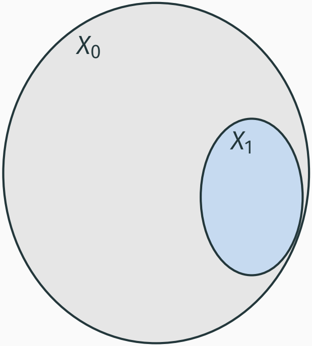


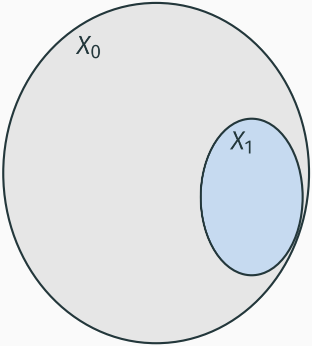
$r_0 :$ Configs $\to \mathbb{N}$

- $C \in X_i \wedge C \xrightarrow{*} C' \implies C' \in X_i$
- $X_0 \supseteq \{C : C \text{ is initial and } \varphi(C) = b\}$

X0 contains all initial configs

Correctness certificates

Approach: certify that φ is computed correctly for $b \in \{0, 1\}$



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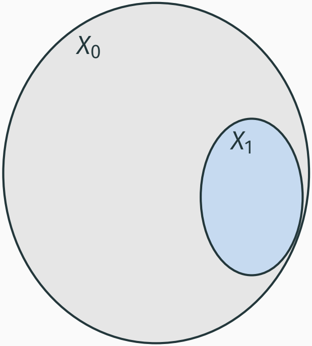
- $X_0 \supseteq \{C : C \text{ is initial and } \varphi(C) = b\}$
- $X_1 \subseteq \{C : \text{opinion}(C) = b\}$

X_1 only contains configs with b-consensus

Correctness certificates

Approach: certify that φ is computed correctly for $b \in \{0, 1\}$



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r_0 is nondecreasing

Correctness certificates

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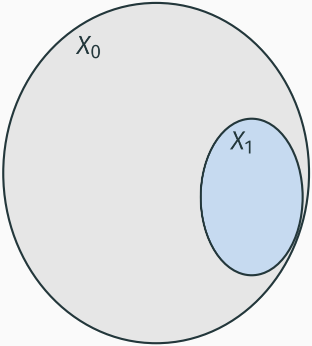

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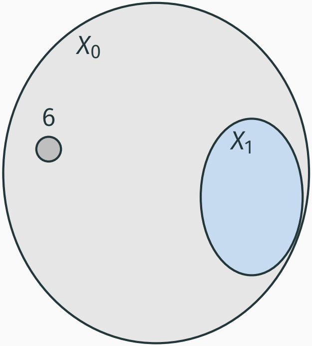
r0 is weakly decreasing

Correctness certificates

Approach: certify that φ is computed correctly for $b \in \{0, 1\}$



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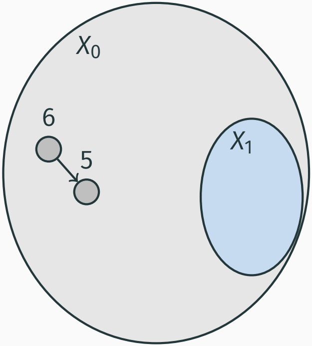

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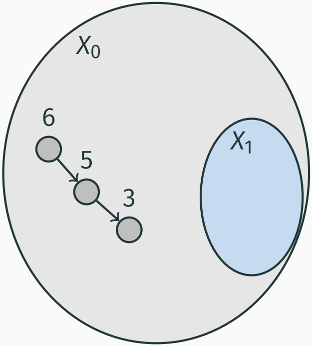

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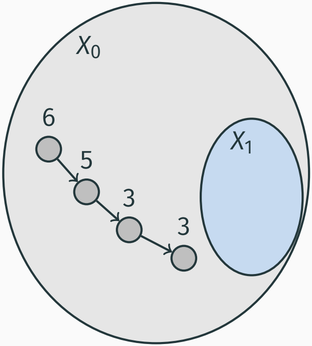

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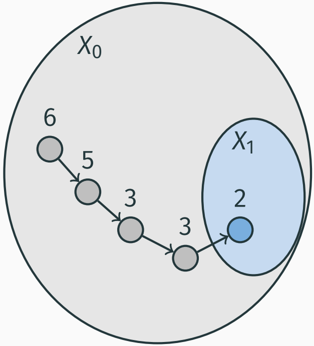

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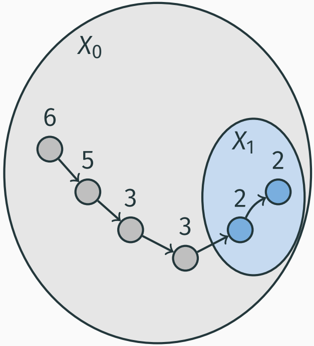

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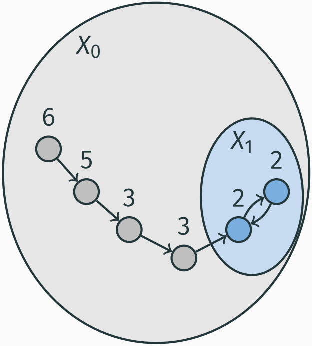

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Stage graphs

Stage graph: same idea with $X_0, X_1, \ldots, X_k$ organized in a DAG

$$
\begin{aligned}
\mathbf{B}\,\mathbf{R} &\to \mathbf{b}\,\mathbf{b}\\
\mathbf{B}\,\mathbf{r} &\to \mathbf{B}\,\mathbf{b}\\
\mathbf{R}\,\mathbf{b} &\to \mathbf{R}\,\mathbf{r}\\
\mathbf{b}\,\mathbf{r} &\to \mathbf{b}\,\mathbf{b}
\end{aligned}
$$

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$$\mathbf{B}\ \mathbf{R} \rightarrow \mathbf{b}\ \mathbf{b}$$
$$\mathbf{B}\ \mathbf{r} \rightarrow \mathbf{B}\ \mathbf{b}$$
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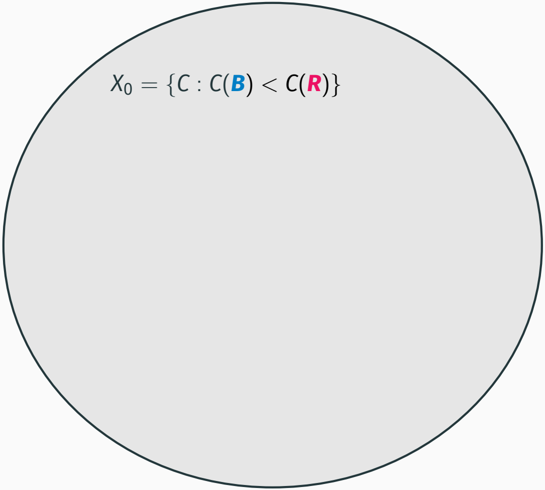

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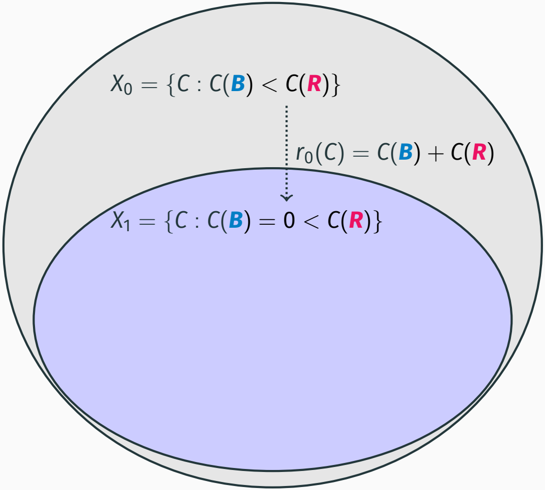
$$\begin{aligned} B\,R &\to b\,b \\ B\,r &\to B\,b \\ R\,b &\to R\,r \\ b\,r &\to b\,b \end{aligned}$$



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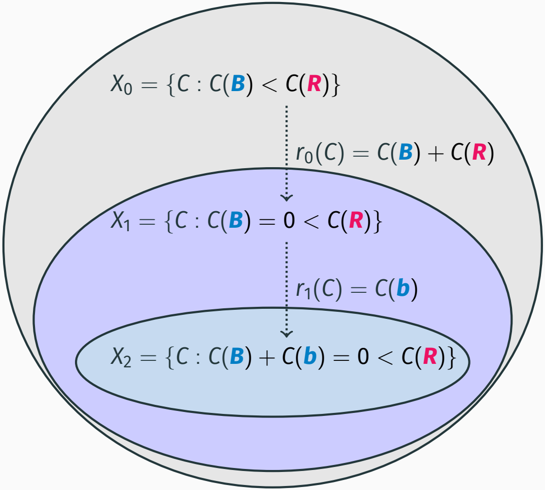
$$B\;R \to b\;b$$
$$B\;r \to B\;b$$
$$R\;b \to R\;r$$
$$b\;r \to b\;b$$



Stage graphs

A stage graph is *Presburger* if

- Each set X_i is Presburger-definable
- Each ranking function r_i is Presburger-definable
- Each r_i can be decreased in at most B_i steps

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Theorem

Every correct protocol has Presburger stage graphs

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Theorem

Every correct protocol has Presburger stage graphs

Computable and checkable in practice with SMT solving!

Demonstration

Expected termination time

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{b}, \mathbf{b}$

$\mathbf{B}, \mathbf{r} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{b} \mapsto \mathbf{R}, \mathbf{r}$

$\mathbf{b}, \mathbf{r} \mapsto \mathbf{b}, \mathbf{b}$

Correctly computes predicate $\#\mathbf{B} \geq \#\mathbf{R}$

...but how fast?

Expected termination time

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{b}, \mathbf{b}$

$\mathbf{B}, \mathbf{r} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{b} \mapsto \mathbf{R}, \mathbf{r}$

$\mathbf{b}, \mathbf{r} \mapsto \mathbf{b}, \mathbf{b}$

Correctly computes predicate $\#\mathbf{B} \geq \#\mathbf{R}$

...but how fast?

- **Natural to look for fast protocols**
- **Bounds on expected termination time useful since generally not possible to know whether a protocol has stabilized**

Expected termination time

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{b}, \mathbf{b}$

$\mathbf{B}, \mathbf{r} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{b} \mapsto \mathbf{R}, \mathbf{r}$

$\mathbf{b}, \mathbf{r} \mapsto \mathbf{b}, \mathbf{b}$

Correctly computes predicate $\#\mathbf{B} \geq \#\mathbf{R}$

...but how fast?

Theorem **Angluin *et al.* PODC'04**

Every Presburger-definable predicate is computable by a protocol with expected termination time $\in \mathcal{O}(n^2 \log n)$

Expected termination time

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{b}, \mathbf{b}$$
$$\mathbf{B}, \mathbf{r} \mapsto \mathbf{B}, \mathbf{b}$$
$$\mathbf{R}, \mathbf{b} \mapsto \mathbf{R}, \mathbf{r}$$
$$\mathbf{b}, \mathbf{r} \mapsto \mathbf{b}, \mathbf{b}$$

Simulations show that it is slow when **R** has slight majority:

Steps	Initial configuration
100000	{B: 7, R: 8}
7	{B: 3, R: 12}
27	{B: 4, R: 11}
100000	{B: 7, R: 8}
3	{B: 13, R: 2}

Expected termination time

$$\begin{aligned}
\mathbf{B}, \mathbf{R} &\mapsto \mathbf{T}, \mathbf{t} & X, y &\mapsto X, x \quad \text{for } x, y \in \{\mathbf{b}, \mathbf{r}, \mathbf{t}\} \\
\mathbf{B}, \mathbf{T} &\mapsto \mathbf{B}, \mathbf{b} \\
\mathbf{R}, \mathbf{T} &\mapsto \mathbf{R}, \mathbf{r} \\
\mathbf{T}, \mathbf{T} &\mapsto \mathbf{T}, \mathbf{t}
\end{aligned}$$

$$O(\mathbf{B}) = O(\mathbf{b}) = O(\mathbf{T}) = O(\mathbf{t}) = 1$$
$$O(\mathbf{R}) = O(\mathbf{r}) = 0$$

Alternative protocol with explicit ties

Expected termination time

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t} \qquad X, y \mapsto X, x \quad \text{for } x, y \in \{\mathbf{b}, \mathbf{r}, \mathbf{t}\}$$

$$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$$

$$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$$

$$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$$

Is it faster?

$$O(\mathbf{B}) = O(\mathbf{b}) = O(\mathbf{T}) = O(\mathbf{t}) = 1$$

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Alternative protocol with explicit ties

Expected termination time

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$

$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$

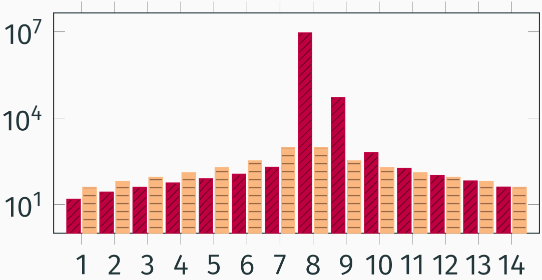
$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$

$X, y \mapsto X, x$ for $x, y \in \{\mathbf{b}, \mathbf{r}, \mathbf{t}\}$

Is it faster?

Yes, for size 15...

expected number of steps to stable consensus



number of agents initially in state **R**

Expected termination time

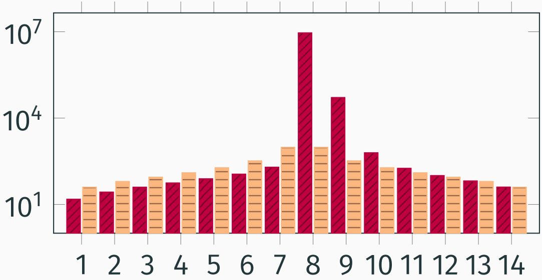
$$\begin{aligned} \mathbf{B}, \mathbf{R} &\mapsto \mathbf{T}, \mathbf{t} \\ \mathbf{B}, \mathbf{T} &\mapsto \mathbf{B}, \mathbf{b} \\ \mathbf{R}, \mathbf{T} &\mapsto \mathbf{R}, \mathbf{r} \\ \mathbf{T}, \mathbf{T} &\mapsto \mathbf{T}, \mathbf{t} \end{aligned}$$

$$X, y \mapsto X, x \quad \text{for } x, y \in \{\mathbf{b}, \mathbf{r}, \mathbf{t}\}$$

Obtained using PRISM

Clément et al. ICDCS'11, Offtermatt '17

expected number of steps to stable consensus



number of agents initially in state **R**

Expected termination time

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$$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$$
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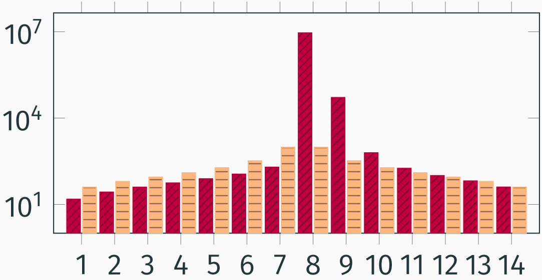
$$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$$

$$X, y \mapsto X, x \quad \text{for } x, y \in \{\mathbf{b}, \mathbf{r}, \mathbf{t}\}$$

Goal: analyze time for all sizes

expected number of steps to stable consensus



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Expected termination time: formal definition

Random variable $Steps_X$:

assigns to each run σ the smallest k s.t. $\sigma_k \in X$, otherwise ∞

Expected termination time: formal definition

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Maximal expected termination time

We are interested in $\mathit{time}\colon \mathbb{N} \to \mathbb{N}$ where

$$\mathit{time}(n) = \max\{\mathbb{E}_C[\mathit{Steps}_{\text{Stable}}] : C \text{ is initial and } |C| = n\}$$

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X, y &\mapsto X, x
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$$X, y \mapsto X, x$$

$$\{C : C(\mathbf{B}) = C(\mathbf{R}) \wedge C(\mathbf{B}) + C(\mathbf{R}) + C(\mathbf{T}) > 0\}$$

Expected termination time: stage graphs

$$
\begin{aligned}
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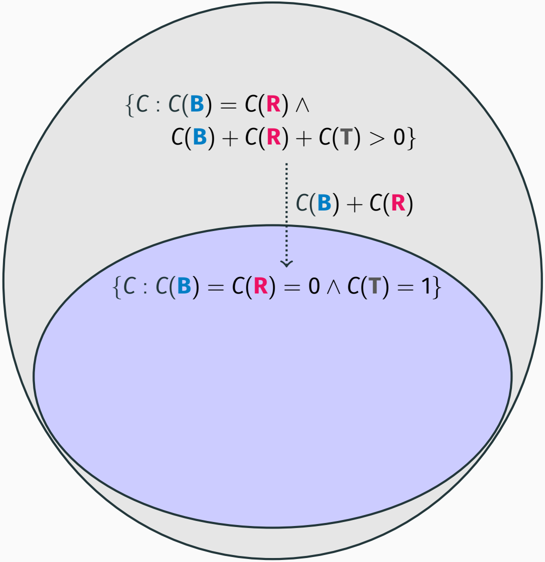

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$$
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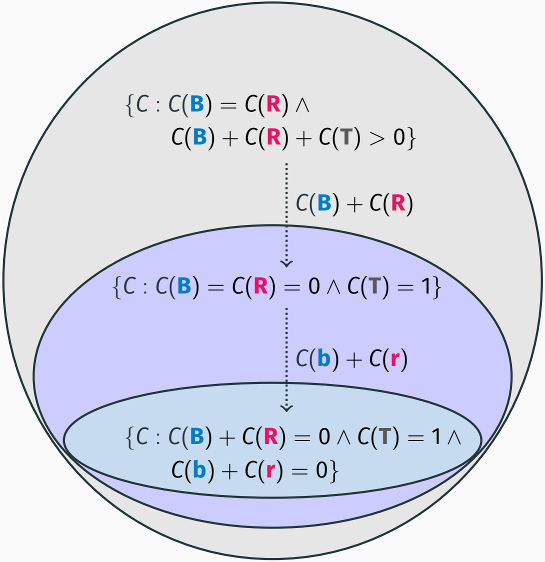


Expected termination time: stage graphs

$$\mathbf{B}, \mathbf{R} \mapsto \mathbf{T}, \mathbf{t}$$

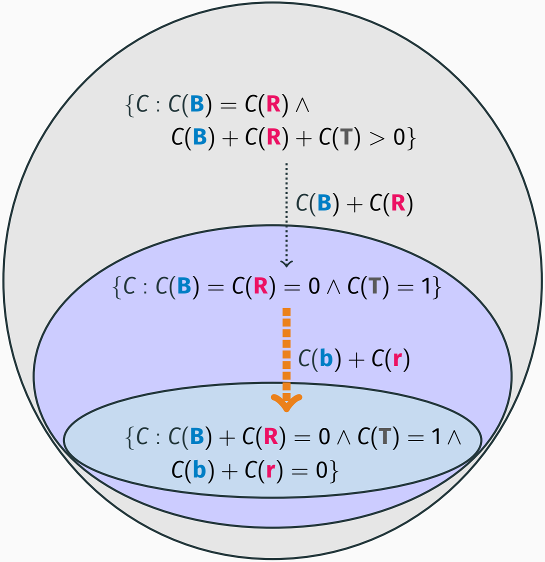
$$\mathbf{B}, \mathbf{T} \mapsto \mathbf{B}, \mathbf{b}$$

$$\mathbf{R}, \mathbf{T} \mapsto \mathbf{R}, \mathbf{r}$$

$$\mathbf{T}, \mathbf{T} \mapsto \mathbf{T}, \mathbf{t}$$

$$X, y \mapsto X, x$$

$$\begin{aligned}\mathbb{E}_C[Steps_{C(\mathbf{b})+C(\mathbf{r})=0}] &\leq \sum_{i=1}^{C(\mathbf{b})+C(\mathbf{r})} \frac{n^2}{2 \cdot C(\mathbf{T}) \cdot i} \\ &\leq \sum_{i=1}^{n} \frac{n^2}{i} \\ &\leq \alpha \cdot n^2 \cdot \log n\end{aligned}$$



In practice, able to report:

$$\mathcal{O}(n^2),\ \mathcal{O}(n^2 \log n),\ \mathcal{O}(n^3),\ \mathcal{O}(n^c),\ \mathcal{O}(2^n)$$

Demonstration

Population protocols analyzable automatically:

- Verification + explanation of correctness
- Bounds on expected termination time
- Tool support

Conclusion: future work

- Asymptotic *lower* bounds on expected termination time?
- Verification of extensions of the model?
- Quantitative model checking?

Thank you!